HEAT KERNEL EXPANSIONS IN VECTOR BUNDLES
by
Martin N. Ndumu
Department of Mathematics and Computer Science
University of Maryland Eastern Shore
Princess Anne, MD 21853
E-mail: mndumu@umes.edu
AMS Mathematics Subject Classification (2000): 58J35, 58J65

Abstract
Let M be a complete connected smooth (compact) Riemannian manifold of dimension n. Let \( \Pi : V \to M \) be a smooth vector bundle over M. Let \( L = \frac{1}{2}\Delta + b \) be a second order differential operator on M, where \( \Delta \) is a Laplace-Type operator on the sections of the vector bundle V and b a smooth vector field on M. Let \( k_t(\cdot, \cdot) \) be the heat kernel of V relative to L. In this paper we will derive an exact and an asymptotic expansion for \( k_t(x, y_0) \) where \( y_0 \) is the center of normal coordinates defined on M, x is a point in the normal neighborhood centered at \( y_0 \). The leading coefficients of the expansion are then computed at \( x = y_0 \) in terms of the linear and quadratic Riemannian curvature invariants of the Riemannian manifold M, of the vector bundle V, and of the vector bundle section \( \phi \) and its derivatives.

We end by comparing our results with those of previous authors (I. Avramidi, P. Gilkey, and McKean-Singer).

Key Words and Phrases
Riemannian manifold, heat kernel, exponential map, normal coordinates, normal neighborhood, geodesic, Brownian Riemannian bridge process, vector bundle, curvature, curvature form, connection form, Laplace-Type operator, expansion coefficients.

1. Introduction

Let M be a complete connected n-dimensional Riemannian manifold. In this paper \( \Delta^0 \) will denote the usual Laplace-Beltrami operator on smooth functions defined on M and b is a smooth vector field on M. \( \Delta \) will denote a Laplace-Type operator defined on smooth sections of a vector bundle V. \( k_t(\cdot, \cdot) \) will denote the heat kernel on the vector bundle V relative to \( L = \frac{1}{2}\Delta + b \).

In section 2 we derive the vector bundle results needed to compute the leading coefficients of the expansion of the heat kernel \( k_t(x, y_0) \) at \( x = y_0 \).

In section 3 we define normal coordinates and then use them to derive two partial differential equations: the first one (Lemma 5) is used to prove a Feynman-Kac type formula (Theorem 1) which in turn is used to prove the generalized Elworthy-Truman heat kernel formula on vector bundles (Theorem 2), and the second one (Theorem 4) plays a vital role in the proofs of the exact expansion formula (Theorem 6) as well as the asymptotic expansion formula (Theorem 7).

In section 4 we give the differential geometric tools necessary for the computation of the leading coefficients of the expansion. These consist of the expansions of the components \( g_{\alpha \beta} \) of the metric tensor defined in normal coordinates and the volume change factor \( \vartheta_{y_0} \) under the exponential map \( \exp_{y_0} : T_{y_0} M \to M \). The techniques we use for these expansions will closely follow those of A. Gray [1] and [2].

The leading coefficients are computed in section 5 and compared to those obtained by previous authors (I. Avramidi [3], P. Gilkey [4], [5], McKean-Singer [6]).
Heat kernel expansions are abundant in the literature. They are useful in quantum field theory, in spectral geometry, in the heat kernel proofs of the index theorem and more.

2. SOME VECTOR BUNDLE CALCULUS

2.1. Connection on a Vector Bundle. Below we first give some definitions and some useful formulae on vector bundles.

\( \nabla : \Gamma(V) \to \Gamma(T^*M \otimes V) \) will be a connection on the vector bundle \( V \) over \( M \), i.e. \( \nabla \) takes smooth sections of \( V \) to smooth sections of the (tensor product) vector bundle \( T^*M \otimes V \). The connection must satisfy the Leibnitz rule: for \( f \in C^\infty(M) \), \( \phi \in \Gamma(V) \),

\[ \nabla(f \phi) = df \otimes \phi + \nabla \phi \]

We will assume that \( \nabla \) is a metric connection, in the sense that for \( \phi_1, \phi_2 \in \Gamma(V) \), we have:

\[ d(\phi_1, \phi_2) = (\nabla \phi_1, \phi_2) + (\phi_1, \nabla \phi_2) \]

where \( d \) is the exterior derivative and \( (, ) \) is the inner product on the fibers of \( V \). The Christoffel symbols of \( \nabla \) will be denoted \( (A^k)_{ij} \) and \( (\Gamma^k_{ij}) \) will denote the Christoffel symbols of the Levi-Civita connection relative to \( TM \).

Let \( \partial_i \equiv \frac{\partial}{\partial x^i} \) be a coordinate frame field on \( TM \). For a function \( f \) on \( M \) (resp. a section \( \phi \) of \( V \)) \( f_i \) (resp. \( \phi_i \)) will sometimes be used to denote \( \frac{\partial}{\partial x_i} \) (resp. \( \nabla_i \phi \)).

At the center of normal coordinates \( y_0 \), \( \phi_i \) reduces to \( \frac{\partial \phi}{\partial y_i} \).

An operator of Laplace-Type \( \Delta \) is defined as follows:

\[ (\Delta \phi) = \frac{1}{2} \Delta \phi = \frac{1}{2} \text{trace} \nabla^2 \phi + W(\phi) \]

where \( W \) is the Weitzenbóckian. It is an endomorphism \( W \in \text{End}(V) \) and,

\[ \text{trace} \nabla^2 \phi = g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \Gamma^k_{ij} \nabla_{\partial_k})\phi \]

Set:

\[ \phi_{ij} = \text{trace} \nabla^2 \phi = g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \Gamma^k_{ij} \nabla_{\partial_k})\phi \]

Then, for an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_x M \),

\[ \text{trace} \nabla^2 \phi = (\nabla_{e_i} \nabla_{e_i} - \Gamma^k_{ij} \nabla_{e_k})\phi = \phi_{ii} \]

Since \( \{\partial_1, \ldots, \partial_n\} \) is orthonormal at \( y_0 \), and since \( \Gamma^k_{ij}(y_0) = 0 \) then,

\[ \Delta \phi(y_0) = \frac{1}{2} \{ \phi_{ii} + W(\phi) \}(y_0) = \frac{1}{2} \left\{ \frac{\partial^2 \phi}{\partial y_i \partial y_i} + W(\phi) \right\}(y_0) \]

2.2. Connection Forms and Curvature Forms. The notion of the connection \( \nabla \) defined above can be extended to an operator \( \hat{\nabla} \) that is extended to differential forms (see for example Berline, Getzler and Vergne [7], p. 21; Lawson and Michelsohn, [8] Proposition (4.6), p. 105; Milnor and Stasheff [9], p. 292; Morita [10], Proposition (5.24), p. 193) as follows:

\[ \hat{\nabla} : \Gamma(A^k(M) \otimes V) \to \Gamma(A^{k+1}(M) \otimes V) \]

is defined by:

\[ \hat{\nabla}(\theta \otimes s) = d\theta \otimes s + (-1)^k \theta \wedge \nabla s \]

For \( k = 0 \), \( \hat{\nabla} = \nabla \) and \( \theta = f \) where \( f \) is now a real-valued function.

Let \( k = 1 \) and set \( \Omega = \hat{\nabla} \circ \nabla \). Then it can be shown (see for example [11], Theorem (3.1.2), p. 108 or [8], Proposition (4.6), p. 105 or [9], Lemma (4), p. 292 or [10], Proposition (5.24), p. 193) that \( \Omega \) defined above coincides with the curvature tensor of the connection \( \nabla \). The composition \( \Omega = \hat{\nabla} \circ \nabla \) defines a smooth section of the vector bundle \( \text{Hom}(V, A^2(M) \otimes V) \cong A^2(M) \otimes \text{Hom}(V, V) \).
Let $\Lambda$ be the connection form of the vector bundle $V$. Then $\Lambda$ is an $\text{End}(V)$--valued 1--form, and $\Omega$ is an $\text{End}(V)$--valued 2--form. They are related by the E. Cartan (second) structure equation as follows:

\[(2.5) \quad \Omega = d\Lambda + \Lambda \wedge \Lambda\]

Then setting:

\[\Lambda = \Lambda_i dx^i \quad \text{and} \quad \Omega = \frac{1}{2} \sum_{i,j} \Omega_{ij} dx^i \wedge dx^j, \]

where

\[\Lambda_i = \Lambda(\partial_i) \quad \text{and} \quad \Omega_{ij} = \Omega(\partial_i, \partial_j)\]

are smooth sections of $\text{End}(V)$, we see that (2.5) above now becomes:

\[(2.6) \quad n \sum_{i,j} \left( \frac{\partial \Lambda_i}{\partial x^i} + \Lambda_i \Lambda_j \right) dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} \Omega_{ij} dx^i \wedge dx^j, \]

and,

\[(2.7) \quad n \sum_{i,j} \left( \frac{\partial \Lambda_j}{\partial x^j} + \Lambda_j \Lambda_i \right) dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} \Omega_{ij} dx^i \wedge dx^j \]

But (2.7) can be re-written as:

\[(2.8) \quad n \sum_{i,j} - \left( \frac{\partial \Lambda_j}{\partial x^j} + \Lambda_j \Lambda_i \right) dx^i \wedge dx^j = - \frac{1}{2} \sum_{i,j} \Omega_{ij} dx^i \wedge dx^j \]

Adding (2.6) and (2.8) we have,

\[(2.9) (a) \quad n \sum_{i,j} \left( \frac{\partial \Lambda_i}{\partial x^i} - \frac{\partial \Lambda_j}{\partial x^j} + \Lambda_i \Lambda_j - \Lambda_j \Lambda_i \right) dx^i \wedge dx^j = 0 \]

We can then equate the coefficients on both sides of (2.9), noting that they are 0 for $i = j$:

\[(2.9) (b) \quad \frac{\partial \Lambda_i}{\partial x^i} - \frac{\partial \Lambda_j}{\partial x^j} + \Lambda_i \Lambda_j - \Lambda_j \Lambda_i = 0 \quad \text{for all} \quad i, j = 1, ..., n. \]

We note that we cannot equate coefficients in (2.6) for all $i, j = 1, ..., n$ because if $i = j$, then both sides are equal (to 0) but $\frac{\partial \Lambda_i}{\partial x^i} + \Lambda_i \Lambda_i$ will not necessarily be equal to $\Omega_{ii} = 0$. In (2.6) we can equate coefficients only for $i < j$.

We now assume that $y_0$ is the center of a normalized local frame $\mu_1, \ldots, \mu_m$ for $V$.

**Lemma 1**

(i) $\frac{\partial \Lambda_i}{\partial x^i}(y_0) = \frac{1}{2} \Omega_{ij}(y_0)$

(ii) $\frac{\partial^2 \Lambda_i}{\partial x^i \partial x^j}(y_0) = \frac{1}{2} \Omega_{ij}(y_0)$

(iii) $\frac{\partial^2 \Lambda_i}{\partial x^j \partial x^j}(y_0) = 0$

(iv) $\frac{\partial^2 \Lambda_i}{\partial x^i \partial x^j}(y_0) = \frac{1}{2} \Omega_{ij}(y_0)$

(v) $\frac{\partial^2 \Lambda_i}{\partial x^i \partial x^j}(y_0) = 0$

**Proofs:**

The proofs use (2.11.1) of [4]:

(i) This has already been computed in [7] Proposition (1.18), p. 26.

(ii) The expansion formula for $\Lambda_j$ in [7], Proposition (1.18)) or ([12], p. 38), gives: $\Lambda_j(x) = \frac{1}{4} \Omega_{jk} x_k + O(|x|^3)$

Consequently,

$\Lambda^2_j(x) = \frac{1}{4} \Omega_{jk} \Omega_{ji}(y_0)x_k x_i + O(|x|^3),$

and differentiating twice with respect to $x_i$, we have:

$\frac{\partial^2 \Lambda^2_j}{\partial x^i \partial x^i}(x) = \frac{1}{4} \Omega_{jk} \Omega_{ji}(y_0) \delta_{ik} \delta_{ij} + \frac{1}{4} \Omega_{ji} \delta_{ij} + O(|x|)$

Therefore at the center of normal coordinates $y_0$, we have (since $y_0 \equiv 0$):

$\frac{\partial^2 \Lambda^2_j}{\partial x^i \partial x^j}(y_0) = \frac{1}{4} \Omega_{ji}(y_0) + \frac{1}{4} \Omega_{ji}(y_0) = \frac{1}{2} \Omega_{ij}(y_0)$
In order to prove (iii), (vi) and (v) we need to use the full formula for the expansion of \( \Lambda_j \) in the neighborhood of \( y_0 \) given in ([7], Proposition (1.18)).

First we clarify the notation used there.

Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). We define the multi-index:
\[
\alpha = (\alpha_2, \ldots, \alpha_n) \quad \text{where} \quad \alpha_i \in \mathbb{N} = \{0, 1, 2, \ldots, \}
\]
\[
|\alpha| = \alpha_2 + \alpha_3 + \cdots + \alpha_n; \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \quad \text{and} \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}
\]
\[
\partial_i = \frac{\partial}{\partial x_i}; \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}
\]
The Taylor expansion (in the neighborhood of \( y_0 \)) of the 1-form endomorphism \( \Lambda_j \) is given in ([7], Proposition (1.18)) by:
\[
(2.10) \quad \Lambda_j(x) = \frac{1}{2} \sum_{k=1}^{m} \Omega_{ij}(y_0)x_k + \sum_{|\alpha| \geq 2} (\partial^\alpha \Lambda_j)(y_0)x^\alpha/\alpha!
\]
where the partial derivatives \( \partial^\alpha \Lambda_j(y_0) \) at \( y_0 \) of the curvature 1-form \( \Lambda_j \) are related to the partial derivatives \( \partial^\alpha \Omega_{ij}(y_0) \) at \( y_0 \) of the curvature 2-forms \( \Omega_{ij} \) by:
\[
(2.11) \quad \sum_{|\alpha| \geq 1} (|\alpha| + 1)(\partial^\alpha \Lambda_j)(y_0)x^\alpha/\alpha! = \sum_{|\alpha| \geq 0} \sum_{k=1}^{n} (\partial^\alpha \Omega_{kj})(y_0)x_k x^\alpha/\alpha!
\]
To see a direct relationship between the partial derivatives of \( \Lambda_j \) and \( \Omega_{ij} \) at \( y_0 \), we need to equate coefficients of \( x^\alpha \) on both sides of (2.11). Because of the presence of \( x_k \) in the sum on the R.H.S. of (2.11) we need to figure out the coefficient of \( x^\alpha \) with \( |\alpha| = m \) in this sum and equate it to the coefficient of \( x^\alpha \) with \( |\alpha| = m \) on the L.H.S.

First we write out the sum on the R.H.S. of (2.11) in its expanded form:
\[
\sum_{|\alpha| \geq 0} \sum_{k=1}^{n} (\partial^\alpha \Omega_{kj})(y_0)x_k x^\alpha/\alpha! = \sum_{k=1}^{n} x_k \sum_{|\alpha| \geq 0} (\partial^\alpha \Omega_{kj})(y_0) x^\alpha/\alpha! = \sum_{k=1}^{n} x_k \Omega_{ij}(y_0) + \sum_{|\alpha| = 2} (\partial^\alpha \Omega_{kj})(y_0)x^\alpha/\alpha! + \ldots
\]

We see that the coefficient of \( x^\alpha \) with \( |\alpha| = m \) on the L.H.S. of (2.11) is given by (\( |\alpha| + 1)(\partial^\alpha \Lambda_j)(y_0) \frac{1}{\alpha!} \) (with \( |\alpha| = m \)) whereas the coefficient of \( x^\alpha \) with \( |\alpha| = m \) on the R.H.S. is given by \( (\partial^\alpha \Omega_{kj})(y_0) \frac{1}{\alpha!} \) (with \( |\alpha| = m - 1 \)). This is because of the presence of \( x_k \) on the R.H.S.: \( x_k x^\alpha \) (with \( |\alpha| = m - 1 \)) is identical to \( x^\alpha \) (with \( |\alpha| = m \)).

We conclude that,
\[
(2.12) \quad (|\alpha| + 1)(\partial^\alpha \Lambda_j)(y_0) \frac{1}{\alpha!} = (\partial^\alpha \Omega_{kj})(y_0) \frac{1}{\alpha!} \quad \text{(with} \quad |\alpha| = m - 1 \text{)}
\]
This means that in the Taylor expansion of \( \Lambda_j \), the coefficient to the order \( m \) at \( y_0 \), depends only on the coefficient to the order \( m - 1 \) at \( y_0 \) of the Taylor expansion of \( \Omega_{kj} \). We can now prove (iii), (iv) and (v):

**Proof of (iii):**
We consider \( (\partial^\alpha \Lambda_j)(y_0) = \frac{\partial^2 \Lambda_j}{\partial x_i \partial x_k}(y_0) \)
In this case, \( \alpha = (1, 1) \) and hence \( \alpha! = 1!1! = 1 \) and \( |\alpha| = 2 \).
Consequently,
\[
3 \frac{\partial^2 \Lambda_j}{\partial x_i \partial x_k}(y_0) = \frac{\partial \Omega_{kj}}{\partial x_i}(y_0)
\]
and so,
\[
\frac{\partial^3 \Lambda_{ij}}{\partial x_i \partial x_j}(y_0) = \frac{1}{3} \frac{\partial \Omega_{ij}}{\partial x_i}(y_0) \pm \frac{1}{3} \Omega_{ij}(y_0)
\]

Proof of (iv):
We consider the case:
\[
(\partial^3 \Lambda_i)(y_0) = \frac{\partial^3 \Lambda_i}{\partial x_i \partial x_j \partial x_k}(y_0)
\]
We see that,
\[
\alpha = (2, 1, 1). \text{ Hence } \alpha! = 2!1! = 2 \text{ and } |\alpha| = 3
\]
Consequently,
\[
(|\alpha| + 1)(\partial^3 \Lambda_i)(y_0) = \frac{1}{2} \frac{\partial^3 \Lambda_i}{\partial x_i \partial x_j \partial x_k}(y_0)
\]
The other hand,
\[
(\partial^3 \Omega_{ij})(y_0) = \frac{1}{2} \frac{\partial^3 \Omega_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0)
\]
and hence,
\[
2 \frac{\partial^3 \Lambda_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0) = \frac{1}{2} \frac{\partial^3 \Omega_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0).
\]

Proof of (v)
We deduce from (iv) that,
\[
\frac{\partial^3 \Lambda_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0) = \frac{1}{2} \frac{\partial^3 \Omega_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0)
\]
Since \(\Omega_{ij}\) is skew-symmetric in \((i, j)\) (and hence \(\Omega_{jj} = 0\)),
\[
\frac{\partial^3 \Lambda_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0) = \frac{1}{4} \frac{\partial^3 \Omega_{ij}}{\partial x_i \partial x_j \partial x_k}(y_0) = 0
\]

2.3. Local expression of the Laplace-Type Operator. By [11], p. 104 or by
[12], p. 29, the connection \(\nabla\) can locally be decomposed as:
\[
\nabla = d + \Lambda
\]
in the sense that, for a section \(\phi = \phi^i \mu_j\) where \(\mu_1, ..., \mu_m\) is a local frame field
for \(V\), we have:
\[
\nabla(\phi^i \mu_j) = d(\phi^i \mu_j) + \phi^i \Lambda_j \mu_j = \frac{\partial \phi^i}{\partial x_k} dx^k \mu_j + \phi^i d \mu_j + \phi^i \Lambda_k dx^k \mu_j
\]
(since \(d \mu_j = 0\) by the fact that \(\nabla\) is a metric connection). Hence,
\[
\nabla_{\partial_i}(\phi^i \mu_j) = \frac{\partial \phi^i}{\partial x_k} dx^k (\frac{\partial \mu_j}{\partial x_i}) + \phi^i \Lambda_k dx^k \mu_j
\]
\[
= \frac{\partial \phi^i}{\partial x_k} \delta^j_k \mu_j + \phi^i \Lambda_k \delta^j_k \mu_j
\]
\[
= \frac{\partial \phi^i}{\partial x_k} \mu_j + \phi^i \Lambda_k \mu_j = (\frac{\partial}{\partial x_i} + \Lambda_j)(\phi^i \mu_j)
\]
Consequently, setting \(\partial_i = \frac{\partial}{\partial x_i}\), we can write:
\[
\nabla_{\partial_i} = (\frac{\partial}{\partial x_i} + \Lambda_i)
\]

Lemma 2
(i) \(\nabla_{\partial_i} \nabla_{\partial_j} \phi = \left\{ \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial \Lambda_{ij}}{\partial x_i} + \Lambda_j \frac{\partial}{\partial x_j} + \Lambda_i \frac{\partial}{\partial x_j} + \Lambda_i \Lambda_j \right\} \phi\)
(ii) \(L \phi = \frac{1}{2} \Delta \phi(y_0) = \frac{1}{2} \left\{ g^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial \Lambda_{ij}}{\partial x_i} + \Lambda_j \frac{\partial}{\partial x_j} + \Lambda_i \frac{\partial}{\partial x_j} + \Lambda_i \Lambda_j \right) \right\} \phi
\]
= \(\frac{1}{2} \left\{ g^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial \Lambda_{ij}}{\partial x_i} + \Lambda_j \frac{\partial}{\partial x_j} + \Lambda_i \frac{\partial}{\partial x_j} + \Lambda_i \Lambda_j \right) \right\} \phi
\]
= \(\frac{1}{2} \left\{ g^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial \Lambda_{ij}}{\partial x_i} + \Lambda_j \frac{\partial}{\partial x_j} + \Lambda_i \frac{\partial}{\partial x_j} + \Lambda_i \Lambda_j \right) \right\} \phi
\]
where \(\Delta^o\) is the Laplacian on functions.
(iv) When $y_0$ is the center of a normalized local frame $\mu_1, ..., \mu_m$ for $V,$ as in lemma (2.1.1) of [4], then:

$$L(f \phi)(y_0) = f(y_0)L \phi(y_0) + (L(\phi))(y_0)\phi(y_0) + f_{ij}(y_0)\phi_{ij}(y_0)$$

**Proof**

(i) $\nabla_{\partial_i} \phi = \left( \frac{\partial}{\partial x_i} + \Lambda_j \right) \phi = \frac{\partial \phi^k}{\partial x_i} \mu_k + \Lambda_j \phi^k \mu_k = \left( \frac{\partial \phi^k}{\partial x_i} + \Lambda_j \phi^k \right) \mu_k$

Consequently, we have:

$$\nabla_{\partial_i} \nabla_{\partial_j} \phi = \nabla_{\partial_i} \left( \frac{\partial \phi^k}{\partial x_j} + \Lambda_j \phi^k \right) \mu_k = \left( \frac{\partial^2 \phi^k}{\partial x_i \partial x_j} + \Lambda_j \frac{\partial \phi^k}{\partial x_i} + \Lambda_i \frac{\partial \phi^k}{\partial x_j} + \Lambda_i \Lambda_j \phi^k \right) \mu_k$$

(ii) is immediate from (2.1) and (i) above.

(iii) is just Calculus.

(iv) Since $\nabla_{\partial_i} = \partial_i + \Lambda_i$ and $\Lambda_i(y_0) = 0$ by lemma (2.1.1) of [4], then since $y_0$ is also the center of Fermi coordinates at which $g^{ij} = \delta^{ij},$ we have:

$$\frac{1}{2}g^{ij}(y_0) \left( \frac{\partial}{\partial x_i} \nabla_{\partial_j} + \frac{\partial}{\partial x_j} \nabla_{\partial_i} \right) \phi \right) \right) (y_0)$$

$$= \frac{1}{2}g^{ij}(y_0) \left( \frac{\partial}{\partial x_i} \left( \frac{\partial \phi^k}{\partial x_j} + \Lambda_j \phi^k \right) + \frac{\partial}{\partial x_j} \left( \frac{\partial \phi^k}{\partial x_i} + \Lambda_i \phi^k \right) \right) \phi \right) \right) (y_0)$$

$$= \frac{\partial}{\partial x_i}(y_0) \frac{\partial \phi^k}{\partial x_j}(y_0) = f_{ij}(y_0)\phi_{ij}(y_0)$$

From now onwards the frame $\mu_1, ..., \mu_m$ is chosen normalized at $y_0.$

\[\square\]

### 3. Analytic Tools and Some Results

#### 3.1. Normal Coordinates

The expansion theorem uses normal coordinates and so we introduce them here.

Normal coordinates are used to describe the geometry of a manifold in the neighborhood of a point and they are well known. They are defined using the exponential map of the tangent bundle.

Let $\exp_{y_0} : T_{y_0} M \to M$ be the exponential map (based at $y_0$) of the tangent bundle. Then we can define normal coordinates as follows:

**Definition** (See for example [2])

Let $E_1, ..., E_n$ be orthonormal sections of the tangent bundle. Then the normal coordinates $(x_1, ..., x_n)$ at $y_0$ relative to $E_1, ..., E_n$ are defined by:

\[
(3.1) \quad x_j(\exp_{y_0}(\sum_{i=1}^{n} t_i E_i)) = t_j \quad \text{for } j = 1, ..., n
\]

Here, $t_1, ..., t_n$ are chosen small enough so that: $\sum_{i=1}^{n} t_i E_i \in B_0,$ where $B_0$ is defined below in (3.2).

The exponential map of the tangent bundle $\exp_{y_0} : T_{y_0} M \to M$ maps a neighborhood $B_0$ of the zero vector in $T_{y_0} M$ diffeomorphically onto a neighborhood $M_0$ of $y_0$ in $M$:

\[
(3.2) \quad M_0 = \exp_{y_0}(B_0)
\]

We call $(\exp_{y_0}^{-1}, M_0)$ a geodesic chart in $M$ based at $y_0.$
Let $\gamma : [0,1] \rightarrow M$ defined by $\gamma(t) = \exp_{y_0}(tv)$. Then $\gamma$ is the unique minimal geodesic such that, $\gamma(0) = y_0$, $\gamma(1) = \exp_{y_0}(v) = x$, and the derivative at 0 $\gamma'(0) = v$.

For the expansion we need the terms defined in (3.3) below:

Let,

$$\Phi_{y_0}(x) = \exp \left\{ \int_0^1 <b(\gamma(s))', \gamma'(s)>ds \right\}$$

where $b$ is the vector field given in the definition of $L = \frac{1}{2} \Delta + b$ and $\gamma$ is the unique minimal geodesic from $x \in M_0$ to $y_0$ in time 1 (the reverse of the one defined above). Let $(g_{\alpha\beta})$ $\alpha, \beta = 1, ..., n$ be the components of the metric tensor field defined by the normal coordinates $x_1, ..., x_n$. The infinitesimal volume change factor,

$$\#_{y_0} : M_0 \rightarrow \mathbb{R}$$

is defined by:

$$\#_{y_0}(x) = T_v \exp_{y_0}$$

where $x = \exp_{y_0} v$.

It is well known that:

$$\#_{y_0}(x) = \sqrt{\det(g_{\alpha\beta}(x))}$$

Set,

$$\Psi(x) = \Phi_{y_0}(x) - \frac{1}{2}$$

and,

$$q_t(x,y_0) = (2\pi t)^{-\frac{n}{2}} \Psi(x) \exp \left\{ -\frac{d(x,y_0)^2}{2t} \right\}$$

### 3.2. Exact Formula for the Heat Kernel.

For the rest of what follows we will assume that $M$ is a compact Riemannian manifold. This restriction is just for convenience. It can be replaced by requiring the smooth sections of vector bundles to have compact support in a non-compact $M$.

Recall that by (2.1) above the Laplace-Type operator $\Delta$ is related to the connection operator $\nabla$ on the vector bundle $V$ by:

$$\Delta = \text{trace}\nabla^2 + W.$$ 

where the Weitzenböck term $W$ is a smooth section of $\operatorname{End}(V)$.

A heat kernel on a vector bundle $V$ is an operator:

$$k_t(x,y) \in \operatorname{Hom}(V_y, V_x)$$

where $V_x$ is a fiber of the vector bundle $V$ which serves as the fundamental solution to the heat equation on the vector bundle $V$:

$$\frac{\partial \phi_t}{\partial t} = \frac{1}{2} \text{trace}\nabla^2 \phi_t + W(\phi_t) + \nabla \phi_t(b)$$

(initial condition)

where $b$ is the smooth vector field on $M$ and $\phi_t$ is a time dependent smooth section of the vector bundle $V$. Then the solution of the above heat equation is given by:

$$\phi_t(x) = \int_M k_t(x, y) \phi(y) v_M(dy)$$

The heat kernel on a vector bundle over a Riemannian manifold $M$ can be derived using normal coordinates along the same lines as in the scalar case (Theorem (4.8) of [13]). For a non-compact manifold we take a section of compact support in $M_0$

For $t \geq s \geq 0$, define the operators $P_t$ (semigroup) and $Q(t,s)$ on $\Gamma(V)$ by:

$$P_t \phi(x) = \int_M (k_t(x,z)\phi(z)) v_M(dz)$$

$$Q(t,s) \phi(x) = q_t(x,y_0)^{-1} P_{t-s}(q_t(-y_0)\phi)$$

Then,

$$Q(t,t-s) \phi(x) = q_t(x,y_0)^{-1} P_{t-s}(q_t(-y_0)\phi)$$

$$= q_t(x,y_0)^{-1} \int_M q_t(z,y_0)(k_{t-s}(x,z)) \phi(z) v_M(dz)$$
The use of the operators $Q(t,s)$ on sections of vector bundles will prove to be very useful in obtaining both a Feynman-Kac-type formula and an exact and asymptotic expansion formulae for the generalized heat kernel on vector bundles similar to the case for the generalized scalar heat kernel in [13].

First we give some preliminary lemmas which are needed to prove the theorems:

**Lemma 3**

Let $\phi$ be a section of $V$ with support in $M_0$ where $(\exp_{y_0}^{-1}, M_0)$ is a geodesic chart in $M$ based at $y_0$ defined in (3.2) above. Then,

\[
\lim_{s \uparrow t}(Q(t,t-s) \phi)(x) = q_t(x,y_0)^{-1}k_t(x,y_0)\phi(y_0)
\]

**Proof**

From (3.14), we have:

\[
(Q(t,t-s) \phi)(x) = q_t(x,y_0)^{-1}P_{(t-s)}(-y_0)\phi(x) = q_t(x,y_0)^{-1}\int_{M_0} q_{t-s}(z,y_0)k_s(x,z)\phi(z)\nu_M(dz)
\]

where $\phi$ has compact support in $M_0$.

Making the change of variable $z = \exp_{y_0} v$ (noting that $\phi_z$ depends smoothly on $z$) and recalling that $\exp_{y_0}(B_0) = M_0$,

\[
(Q(t,t-s) \phi)(x) = q_t(x,y_0)^{-1} \int_{B_0} q_{t-s}(\exp_{y_0} v,y_0)k_s(x,\exp_{y_0} v)\phi(\exp_{y_0} v)\vartheta_{y_0}(v)dv
\]

Setting $t-s = r$ and $v = \sqrt{rw}$, then (3.15) becomes:

\[
\int_{B_0} q_r(\exp_{y_0} \sqrt{rw},y_0)k_s(x,\exp_{y_0} \sqrt{rw})\phi(\exp_{y_0} \sqrt{rw})\vartheta_{y_0}(\sqrt{rw})r^{\frac{p}{2}}dw
\]

We have that,

\[
q_r(\exp_{y_0} \sqrt{rw},y_0)r^{\frac{p}{2}} = (2\pi)^{-\frac{p}{2}}\Psi(x)\exp\left\{-\frac{\|w\|^2}{2}\right\}r^{\frac{p}{2}} = (2\pi)^{-\frac{p}{2}}\Psi(x)\exp\left\{-\frac{\|w\|^2}{2}\right\}
\]

and $\exp_{y_0} 0 = y_0$ and $\vartheta_{y_0}(0) = 1$, and $\int_{B_0} \exp\left\{-\frac{\|w\|^2}{2}\right\}dw = (2\pi)^{\frac{p}{2}}$.

Taking limits as $r \uparrow 0$ (which is equivalent to $s \uparrow t$) on both sides of (3.16), we have:

\[
\lim_{s \uparrow t}(Q(t,t-s) \phi)(x) = k_t(x,y_0)\phi(y_0)
\]

and so,

\[
\lim_{s \uparrow t}(Q(t,t-s) \phi)(x) = q_t(x,y_0)^{-1}k_t(x,y_0)\phi(y_0)
\]

Here we begin the process of heat kernel expansions on vector bundles. The first ingredient is the partial differential equation below derived by a direct computation in [13], Lemma (4.2)

**Lemma 4**

\[
\frac{\partial k_t}{\partial t}(x,y_0) = Lk_t(x,y_0) - \frac{\psi}{\psi}(x)k_t(x,y_0)
\]

where,

\[
L = \frac{1}{2}\Delta + b
\]

and where, $b$ is a smooth vector field on $M$ and $\Psi$ is defined in (3.6).

**Proof**

This lemma is proved in Lemma (4.2) of [13] by a direct computation and does not involve the use of the operators $Q(t,s)$.

The above lemma is used in the proof of the next lemma which, in turn will be used to prove Theorem 1 below.
Lemma 5
\[
\frac{\partial}{\partial t} (Q(t,s) \phi) = (L + \nabla \log q_t (-y_0) + \frac{Lq_t}{V})(Q(t,s) \phi)
\]
where,
\[
L = \frac{1}{2} \Delta + b
\]
and where,
\[
\Delta \text{ is a Laplace-Type operator and } b \text{ is a smooth vector field on } M.
\]
Proof
We will closely follow the proof of Lemma (4.3) in [13], but noting that the operator \( \Delta \)
there is the scalar Laplacian (on functions) and \( \Delta \) here is a Laplace-Type operator.

From the definition of \( Q(t,s) \phi \),
\[
\frac{\partial}{\partial t} (Q(t,s) \phi) = \frac{\partial}{\partial t} (q_t (-y_0)^{-1} P_{t,s} (q_t (-y_0) \phi))
\]
\[
= - q_t (-P)^{-1} \frac{\partial}{\partial t} (q_t (-y_0) P_{t,s} (q_t (-y_0) \phi)) + q_t (-y_0)^{-1} . \frac{\partial}{\partial t} (P_{t,s} (q_t (-y_0) \phi))
\]
\[
= I_1 + I_2, \text{ where,}
\]
\[
I_1 = - q_t (-y_0)^{-1} \frac{\partial}{\partial t} (P_{t,s} (q_t (-y_0) \phi))
\]
\[
= q_t (-y_0)^{-1} L (P_{t,s} (q_t (-y_0) \phi)) \text{ by Theorem (4.3) of [14]}
\]
\[
= q_t (-y_0)^{-1} L \{ q_t (-y_0)^{-1} P_{t,s} (q_t (-y_0) \phi) \} \cdot q_t (-y_0)
\]
\[
= q_t (-y_0)^{-1} L \{ (Q(t,s) \phi) q_t (-y_0) \}
\]
\[
= q_t (-y_0)^{-1} \left[ L((Q(t,s) \phi)) q_t (-y_0) + (Q(t,s) \phi) L(q_t (-y_0)) \right]
\]
\[
= L((Q(t,s) \phi)) + (Q(t,s) \phi) L(q_t (-y_0)) + < \nabla q_t (-y_0), \nabla (Q(t,s) \phi)> >
\]
\[
= L((Q(t,s) \phi)) + (Q(t,s) \phi) L(q_t (-y_0)) + < \nabla q_t (-y_0), \nabla (Q(t,s) \phi)>
\]
\[
= \left\{ L + \nabla \log q_t (-y_0) + \frac{Lq_t}{q_t (-y_0)} \right\} (Q(t,s) \phi)
\]
Then adding: \( I_1 \) and \( I_2 \) we have:
\[
I_1 + I_2 = \left\{ - \frac{Lq_t}{q_t (-y_0)} + \frac{Lq_t}{V} \right\} . (Q(t,s) \phi) + \left\{ L + \nabla \log q_t (-y_0) + \frac{Lq_t}{q_t (-y_0)} \right\} (Q(t,s) \phi)
\]
Next we prove an important **Feynman-Kac**-type formula from which we deduce the generalized Elworthy-Truman heat kernel formula for vector bundles.

Let \((x^i(s))_{0}^{s} \leq t \leq \zeta \) be the semi-classical Brownian Riemannian bridge process from \( x \) to \( y_0 \) in time \( t \) whose first exit time from the geodesic normal neighborhood \( M_0 \) of \( y_0 \) is \( \zeta \). It is the process with differential generator \( \frac{1}{2} \Delta^0 + b + \nabla \log q_{t-s} (-y_0) \).

We will closely follow the proof of Theorem (4.4) of [13]. However we must use Ito’s formula as given, for example in Theorem (4.3) of [14] or Proposition 1B of [15], p. 83., although Elworthy works there on the dual bundle.

**Theorem 1 (Feynman-Kac Type Formula)**

Let \( \phi \) be a section (with compact support in \( M_0 \)) of the vector bundle \( V \). Then,
\[
(Q(t,s) \phi)(x) = E_x (\chi_{\zeta > s} \phi(x(s)) \exp \left\{ \int_{0}^{s} \frac{Lq_t}{V}(x^i(u))du \right\})
\]
Proof
Define the processes \( z(r) r \geq 0 \) on \( M_0 \) by,
\[ z^i(r) = x^i(t \wedge r) \]
Further define \( y(r) 0 \leq r \leq s \) on \( M_0 \) where \( y:[0,s] \times \Omega \times [t-s,t] \rightarrow M_0 \) by,
\[ y(r) = (t-r, z^i(r), w(r)) \]
where,
\[ w(r) = \exp \left\{ \int_0^r \frac{1}{Y}(z^i(u)du) \right\} = \exp \{ f(r) \} \]
where we set,
\[ f(r) = \int_0^r \frac{1}{Y}(z^i(u)du) \]
and so (differentiating),
\[ w'(r) = \exp \{ f(r) \} f'(r) = \exp \{ f(r) \} \frac{1}{Y}(z^i(r)) = \frac{1}{Y}(z^i(r)w(r)) \]
Consequently,
\[ dw(r) = \frac{1}{Y}(z^i(r)w(r))dr \]
Set,
\[ \phi_\tau (x) = (Q(\tau,t-s)\phi)(x) \]
For \( \tau \geq t \geq s > 0 \) set,
\[ h(\tau,x,w) = \phi_\tau(x)w \]
and so,
\[ (3.18) \quad h(y(r)) = h(t-r, z^i(r), w(r)) = \phi_{t-r}(z^i(r))w(r) \]
Consequently we have:
\[ (3.19) \quad dh(y(r)) = d \phi_{t-r}(z^i(r))w(r) + \phi_{t-r}(z^i(r))dw(r) \]
We note that since \( \phi \) is a smooth section of the vector bundle \( V \), then
\( Q(t-t,s)\phi \) is a time-dependent section of the vector bundle \( V \):
\[ \phi_t(x)= (Q(0,t-s)\phi)(x) \]
for each \( x \in M; \phi_t(x) \) is \( C^1 \) in \( t \geq 0 \) and smooth in \( x \in M \).
We use Ito’s formula for vector bundles as for example in [14] Theorem (4.3) p. 279 or in [15], p. 82 of (although Elworthy works there in the dual bundle) to have for the time-dependent section: \( \phi_t = Q(t,t-s)\phi \):
\[ d \phi_{t-r}(z^i(r)) = \frac{\partial \phi_{t-r}}{\partial r} (z^i(r)dr + \nabla \phi_{t-r}(z^i(r))(u_\tau dB_\tau) + <b(z^i(r)) \]
\[ + \nabla \log q_{t-r}(-P)\nabla \phi_{t-r}(z^i(r))dr> + \frac{1}{2} \text{trace} \nabla^2 \phi_{t-r}(z^i(r))dr + W(\phi_{t-r}(z^i(r)))dr \]
where \( u_\tau \) is the horizontal lift of the process \( z^i(r) \) on the frame bundle \( O(M_0) \).
Consequently (3.19) gives,
\[ (3.20) \quad dh(y(r)) = [ \frac{\partial \phi_{t-r}}{\partial r} (z^i(r))(u_\tau dB_\tau) + <b(z^i(r) \]
\[ + \nabla \log q_{t-r}(-P)\nabla \phi_{t-r}(z^i(r))dr> + \frac{1}{2} \text{trace} \nabla^2 \phi_{t-r}(z^i(r))dr + W(\phi_{t-r}(z^i(r)))dr \]
Since,
\[ \phi_{t-r} = Q(t-r,t-s)\phi \]
we can set \( \lambda = t-r \), and have:
\[ \frac{\partial \phi_{t-r}}{\partial r} = - \frac{\partial \phi_{t-r}}{\partial \lambda}(Q(\lambda,t-s)\phi) \]
\[ = - (L + \nabla \log q_{\lambda}(-P) + \frac{1}{Y})(Q(\lambda,t-s)\phi) \]
by Lemma 5.
\[ = - (L + \nabla \log q_{t-r}(-P) + \frac{1}{Y})(Q(t-r,t-s)\phi) \]
We recall that,
\( \Delta = \text{Trace} \nabla^2 + W \) is an operator of Laplace-Type and \( L = \frac{1}{2} \Delta + b \)
Therefore,
\[ \frac{\partial \phi_{t-r}}{\partial r} = - (\frac{1}{2} \text{Trace} \nabla^2 + W + b + \nabla \log q_{t-r}(-P) + \frac{1}{Y})(Q(t-r,t-s)\phi) \]
Inserting this in the expression for \( dh(y(r)) \) in (3.20) we see that,
\[ (3.21) \quad dh(y(r)) = [ \nabla \phi_{t-r}(z^i(r))(u_\tau dB_\tau) ]w(r) \]
Consequently,
\[(3.22) \quad h(y(s)) = h(y(0)) + \int_0^s \nabla \phi_{t-r}(z') (u_r dB_r) \ w(r) = h(y(0)) + M_s,\]
where,
\[(3.23) \quad M_s = \int_0^s \nabla \phi_{t-r}(z') (u_r dB_r) \ w(r) \text{ is a local martingale.}\]
We need to show that $M_s$ is a martingale. By Theorem (1.3), p.151 of [14],
a local martingale $M_s$ is martingale if $E(\sup_{0 \leq s \leq t} |M_s|) < +\infty$.

On the one hand, since $Q(t-s,t-s)$ is an identity operator then,
\[(3.24) \quad h(y(s)) = \phi_{t-s}(z'(s)) w(s) = (Q(t-s,t-s) \phi)(z'(s)) w(s) = \phi(z'(s)) w(s),\]
is bounded since $\phi$ has compact support in $M_0$ and $\zeta$ is the first exit time of $x'(s)$ from $M_0$. Consequently,
\[\sup_{0 \leq s \leq t} |h(y(s))| \text{ is of finite expectation.}\]

On the other hand since by (3.18)
\[h(y(r)) = \phi_{t-r}(z'(r)) w(r)\]
\[h(y(0)) = \phi_{t}(z'(0)) w(0) = (Q(t,t-s) \phi)(z'(0)) w(0)\]
and so,
\[h(y(0)) = (Q(t,t-s) \phi)(x).\]

Now, $s \rightarrow (Q(t,t-s) \phi)(x)$ is continuous in the half-open interval $[0,t)$ and the limit:
\[\lim_{s \uparrow t} (Q(t,t-s) \phi)(x)\]
exists by Lemma 3 above. Therefore,
\[s \rightarrow (Q(t,t-s) \phi)(x)\]
is continuous in the closed interval $[0,t]$ and so,
\[\sup_{0 \leq s \leq t} (Q(t,t-s) \phi)(x) \text{ is finite and hence is of finite expectation since it is independent chance. Consequently, since } h(y(s)) = h(y(0)) + M_s,\]
\[\sup_{0 \leq s \leq t} |M_s| \text{ is of finite expectation and hence is a martingale.}\]

Since $M_s$ is a martingale: $E_x(M_s) = 0$ and hence,
\[(3.25) \quad E_x(h(y(s))) = E_x(h(y(0))) = (Q(t,t-s) \phi)(x),\]
where $E_x$ is the mathematical expectation relative to Weiner space $(\Omega, \mathcal{F}, P_x)$.

By (3.24) and (3.25), we see that:
\[(3.26) \quad (Q(t,t-s) \phi)(x) = E_x(\phi(x'(s) \zeta)) \text{exp}\left\{ \int_0^s \frac{1}{\Psi}(x'(t)) du \right\}.\]
The theorem is proved.

The formula in (3.26) is a Feynman-Kac-type formula. Compare the R.H.S. of (3.26)
to (320) of [15] and to (B.13) of Appendix B in [22].

Combining Lemma 3 and Theorem 1 we get a formula for heat kernels on vector bundles below:

**Theorem 2 (The generalized Elworthy-Truman heat kernel formula).**

The vector bundle heat kernel $k_t(x,y_0)$ has the property:
\[k_t(x,y_0) \phi(y_0) = q_t(x,y_0) E_x(\chi_{s \geq t} \phi(x'(s))) \text{exp}\left\{ \int_0^s \frac{1}{\Psi}(x'(s)) ds \right\}\]
where, $(x'(s))_{0 \leq s \leq t \land \zeta}$ is the semi-classical Brownian Riemannian bridge from $x$ to $y$ in time $t$ and whose exit time from the geodesic normal neighborhood $M_0$ of $y_0$ is $\zeta$. It is the process with differential generator: $\frac{1}{2} \Delta^x + b + \nabla \log q_{t-s}(-y_0)$ defined above.

**Proof**
Theorem 1 above gives,
\[(Q(t,t-s)\phi)(x) = E_x(\chi_{\mathbb{R}^d} \phi(x^t(s)) \exp \left\{ \int_0^s \frac{1}{2} \nabla \phi(x^t(u)) du \right\})\]
By Lemma 3,
\[\lim_{s \to t}(Q(t,t-s) \phi)(x) = q_t(x,y_0)^{-1}k_t(x,y_0)\phi(y_0),\]
and so,
\[q_t(x,y_0)^{-1}k_t(x,y_0)\phi(y_0) = \lim_{s \to t} E_x(\chi_{\mathbb{R}^d} \phi(x^t(s)) \exp \left\{ \int_0^s \frac{1}{2} \nabla \phi(x^t(u)) du \right\})\]
\[= E_x(\chi_{\mathbb{R}^d} \phi(x^t(t)) \exp \left\{ \int_0^1 \frac{1}{2} \nabla \phi(x^t(s)) ds \right\}),\]
and the theorem is proved.

If inside the expectation above we make the change variable:
\[u = \frac{s}{t},\]
then we have:
\[(3.27) \quad k_t(x,y)\phi(x) = q_t(x,y)E_x(\chi_{\mathbb{R}^d} \phi(x^t(1)) \exp \left\{ t \int_0^1 \frac{1}{2} \nabla \phi(x^t(ut)) du \right\})\]
where \((x^t(ut)) 0 \leq u \leq 1\) is now the bridge process from \(x\) to \(y\) in time \(t\).
This is to be compared to the heat kernel formula of K. D. Elworthy in [15] derived there in
the case of differential forms.
Here we begin the process of heat kernel expansions on vector bundles. First we
derive a partial
differential equation (Theorem 3) below which is a "raw" form of Theorem 4
which in turn plays a
central role in deriving the preliminary expansion theorem (Theorem 5) below.

**Theorem 3**
\[\frac{\partial}{\partial s}(Q(t,t-s)\phi) = (Q(t,t-s)\left(\frac{1}{2}\nabla \phi - \frac{1}{2(t-s)} \nabla d(-,y_0)^2, \nabla \phi\right))\]
where \(\phi\) is a section of the vector bundle \(V\).

**Proof**
\[\frac{\partial}{\partial s}(Q(t,t-s)\phi) = q_t(x,y_0)^{-1} \frac{\partial}{\partial s} P_s(q_t \cdot (-,y_0)\phi))\]
\[= q_t(x,y_0)^{-1} \frac{\partial}{\partial s} \int_M q_t \cdot (z,y_0)k_s(x,z)\phi(z)v_M(dz)\]
Now using Lemma 5 above,
\[\frac{\partial}{\partial s} \int_M q_t \cdot (z,y_0)k_s(x,z)\phi(z)v_M(dz)
= \int_M \frac{\partial}{\partial s} q_t \cdot (z,y_0)k_s(x,z)\phi(z)v_M(dz) + \int_M q_t \cdot (z,y_0) \frac{\partial}{\partial s} k_s(x,z)\phi(z)v_M(dz)
= \int_M [-Lq_t \cdot (z,y_0) + \frac{1}{2} \nabla q_t \cdot (z,y_0)]k_s(x,z)\phi(z)v_M(dz) + \int_M q_t \cdot (z,y_0) L(k_s(x,z))\phi(z)v_M(dz)
= P_s[-Lq_t \cdot (-,y_0)\phi + \frac{1}{2} \nabla q_t \cdot (z,y_0)\phi] + LP_s(q_t \cdot (-,y_0)\phi)
= P_s[-Lq_t \cdot (-,y_0)\phi + \frac{1}{2} \nabla q_t \cdot (z,y_0)\phi](v_0) + P_s L(q_t \cdot (-,y_0)\phi)
\]
by (2.5) and (2.6) of [16]
Now,
\[P_s L(q_t \cdot (-,y_0)\phi) = P_s [L(q_t \cdot (-,y_0)\phi + q_t \cdot (-,y_0)\phi + \nabla q_t \cdot (-,y_0)\phi]]\]
Therefore,
\[\frac{\partial}{\partial s} \int_M q_t \cdot (z,y_0)(k_s(x,z)\phi)v_M(dz)
= P_s[-Lq_t \cdot (-,y_0)\phi + \frac{1}{2} \nabla q_t \cdot (z,y_0)\phi + q_t \cdot (-,y_0)\phi + \nabla q_t \cdot (-,y_0)\phi + \nabla q_t \cdot (-,y_0)\phi]\]
Hence,
\[\frac{\partial}{\partial s}(Q(t,t-s)\phi)\]
\[ \begin{aligned}
q(r,s) &= \frac{1}{2} q_r(z, y_0) \phi + q_s(-y_0) L \phi + \nabla q_r(-y_0) \nabla \phi \\
q(r,s) &= \frac{1}{2} q_r(z, y_0) \phi + q_s(-y_0) L \phi \\
+ q_{r,s}(-y_0) <\nabla \log q_{r,s}(-y_0), \nabla \phi > \\
&= \frac{1}{2} q_r(z, y_0) \phi + q_s(-y_0) L \phi + <\nabla \log q_{r,s}(-y_0), \nabla \phi > \\
&= \frac{1}{2} q_r(z, y_0) \phi + q_s(-y_0) L \phi + <\nabla \log q_{r,s}(-y_0), \nabla \phi > \\
&= (Q(t,s) | \frac{L(q(t))}{q(t)} \phi - \frac{1}{q(t)} <\nabla \log \Psi, \nabla \phi > + <\nabla \log \Psi - \frac{1}{2(t-s)} \nabla d(-y_0)^2, \nabla \phi >) \\
&= (Q(t,s) | \frac{L(q(t))}{q(t)} - \frac{1}{2(t-s)} <\nabla d(-y_0)^2, \nabla \phi >) \\
\end{aligned} \]

We can refine the partial differential equation of the above theorem to obtain theorem 4 below by using the operators \( F(r,s) \) on smooth sections of the vector bundle \( V \).

\( F(r,s) \) is defined as follows: if \( \gamma \) is the minimal geodesic from \( x \in M \) to \( y_0 \) in time \( r \), then \( (F(r,s) \phi)(x) = \phi(\gamma(r,s)) \).

The lemma above and the theorem below were proved in the author's Warwick University Thesis in the case of the trivial bundle (i.e. \( \phi \) is a smooth function on \( M \)). This differential equation is crucial for the derivation of the expansion theorem.

**Theorem 4**

For \( t \geq s > 0 \),
\[
\frac{\partial}{\partial s} (Q(t,s) F(t-s,t-r) \phi) = (Q(t,s) (\frac{L(\Psi F(t-s,t-r) \phi)}{\Psi}))
\]

**Proof**

The proof follows the same lines as the proof of Theorem 3 above, except that \( \phi \) is replaced by \( F(t-s,t-r) \phi \):

\[
\frac{\partial}{\partial s} (q_{r-s}(\cdot, y_0) \phi) \quad \text{above is replaced by:}
\]

\[
(3.28) \quad \frac{\partial}{\partial s} (q_{r-s}(\cdot, y_0) F(t-s,t-r) \phi) = - \frac{\partial}{\partial s} \left( q_{r-s}(\cdot, y_0) F(u,t-r) \phi \big|_{u=t-s} - q_{r-s}(\cdot, y_0) \frac{\partial}{\partial s} F(u,t-r) \phi \big|_{u=t-s} \right)
\]

\[
= (- L q_{r-s}(\cdot, y_0) + \frac{1}{q(t)} q_s(\cdot, y_0). F(u,t-r) \phi \big|_{u=t-s} - q_{r-s}(\cdot, y_0) \frac{\partial}{\partial s} F(u,t-r) \phi \big|_{u=t-s})
\]

\[
= (- L q_{r-s}(\cdot, y_0) + \frac{1}{q(t)} q_s(\cdot, y_0). F(t-s,t-r) \phi \big|_{u=t-s} - q_{r-s}(\cdot, y_0) \frac{\partial}{\partial s} F(u,t-r) \phi \big|_{u=t-s})
\]

Consequently,

\[
\frac{\partial}{\partial s} (Q(t,s) F(t-s,t-r) \phi)(x) \\
(3.29) = Q(t,s) \left[ \frac{L(q(t))}{q(t)} - \frac{1}{2(t-s)} <\nabla d(-y_0)^2, \nabla F(t-s,t-r) \phi > \\
- \frac{\partial}{\partial s} F(u,t-r) \phi \big|_{u=t-s} \right]
\]

By the definition of \( F(u,t-r) \phi(x) \),

\[
\left. \frac{\partial}{\partial s} F(u,t-r) \phi \right|_{u=t-s} = \frac{1}{q(t)} \phi(\gamma(u+t-r)) \big|_{u=t-s}
\]

where \( \gamma \) is now the unique minimal geodesic from \( x \in M_0 \) to \( y_0 \) in time \( u \).

In terms of the exponential map of the tangent bundle:

\[
(3.30) \quad \gamma(t) = \exp_y (1 - \frac{t}{u}) v, \quad \text{where, } v = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}
\]

Then we have in terms of normal coordinates:

\[
(3.31) \quad \gamma(t) = ((1 - \frac{t}{u}) x_1, \ldots, (1 - \frac{t}{u}) x_n)
\]

and hence,

\[
(3.32) \quad \gamma(u+t+r) = ((1 - \frac{u+t+r}{u}) x_1, \ldots, (1 - \frac{u+t+r}{u}) x_n)
\]
Therefore, taking derivatives with respect to \( u \):

\[
(3.32) \quad \frac{d}{du} \phi(\gamma(u-t+r))|_{u=t-s} = \nabla \phi(\gamma(u-t+r))(\dot{\gamma}(u-t+r))|_{u=t-s}
\]

This can be written as:

\[
(3.32) \quad \dot{\gamma}(u-t+r)|_{u=t-s} = \dot{\gamma}(r-s) = -\frac{t-r}{(t-s)^2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}
\]

Consequently we have:

\[
(3.33) \quad \frac{d}{du} \phi(\gamma(u-t+r))|_{u=t-s} = \nabla \phi(\gamma(u-t+r))(\dot{\gamma}(u-t+r))|_{u=t-s}
\]

\[
= -\frac{t-r}{(t-s)^2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \nabla \phi(\gamma(r-s))
\]

On the other hand,

\[
(3.34) \quad \nabla \phi \circ \gamma(u-t+r)|_{u=t-s} = g^{ij} \frac{\partial}{\partial x_j} \phi(\gamma(u-t+r))|_{u=t-s}
\]

\[
= g^{ij} \frac{\partial}{\partial x_j} \phi(\gamma(r-s)) |_{u=t-s}
\]

where \( \gamma \) is now the unique minimal geodesic from \( x \) to \( y_0 \) in time \( t-s \). In local normal coordinates \( \gamma \) can be written as:

\[
\gamma(\lambda) = ((1 - \frac{\lambda}{t-s})x_1, \ldots, (1 - \frac{\lambda}{t-s})x_n),
\]

and:

\[
\frac{\partial}{\partial x_j} \gamma(r-s) = (1 - \frac{\lambda}{t-s}) \frac{\partial}{\partial x_j}
\]

for \( j = 1, \ldots, n \) and hence,

\[
\nabla \phi \circ \gamma(r-s) = g^{ij} \frac{\partial}{\partial x_j} \phi(\gamma(r-s)) \frac{\partial}{\partial x_j} \gamma(r-s) \frac{\partial}{\partial x_i}
\]

\[
= g^{ij} \frac{\partial}{\partial x_j} \phi(\gamma(r-s)) \frac{\partial}{\partial x_i}
\]

and so:

\[
(3.35) \quad \nabla \phi(\gamma(r-s)) = \nabla \phi \circ \gamma(r-s) \frac{\partial}{\partial x_i}
\]

Using (3.35) the last equality in (3.33) becomes:

\[
(3.36) \quad \frac{d}{du} \phi(\gamma(u-t+r))|_{u=t-s} = -\frac{1}{(t-s)^2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}, \nabla F(t-s,t-r) \phi
\]

Set:

\[
\rho^2 = \sum_{j=1}^{n} x_j^2
\]

Then a special case of Lemma (2.1) of [17] gives:

\[
\rho^2 = d(y_0)^2
\]

Also a special case of Lemma (3.1) of [17] (the Generalized Gauss lemma) gives:

\[
\rho \nabla \rho = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}
\]

Consequently,

\[
\nabla \frac{d(-y_0)^2}{2} = \nabla \rho^2 = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \text{ and so (3.36) becomes,}
\]

\[
(3.37) \quad \frac{d}{du} \phi(\gamma(u-t+r))|_{u=t-s} = -\frac{1}{(t-s)^2} \nabla \frac{d(-y_0)^2}{2}, \nabla F(t-s,t-r) \phi
\]

Substituting the R.H.S. of (3.37) for \( \frac{d}{du} \phi(\gamma(u-t+r))|_{u=t-s} \) in (3.29) we have:

\[
\frac{\partial}{\partial \phi} (Q(t,t-s)F(t-s,t-r) \phi(x) = Q(t,t-s) \left[ L(\Psi F(t-s,t-r) \phi) \right]
\]

and the theorem is proved.
3.3. Exact and Asymptotic Expansions

Recall that by (3.7), $\Psi(x) = \partial_{y_0}(x)^{-\frac{1}{2}} \Phi(x)$

where $\Phi_{y_0}$ is defined in (3.3) and $\partial_{y_0}$ is defined in (3.4) and (3.5).

Set,

$$L_\phi \phi = \frac{L(\Psi \phi)}{\Psi}$$

Below are the expansion theorems. These were proved in the case of the trivial bundle in the authors Warwick University Thesis.

**Theorem 5**

Let $\gamma$ be the unique minimal geodesic from $x \in M_0$ to $y_0$ in time $t$. Then for $t \geq s \geq 0, 0 \leq \alpha \leq N$, and smooth $\phi$ with compact support in $M_0$, we have:

$$(3.39) \quad (Q(t, t - s) \phi)(x) = \phi(\gamma(s)) + \sum_{\alpha=1}^{N} a_{\alpha}(s,x,Y_0,\phi)$$

$$+ F_{N+1}(s,x,Y_0,\phi)$$

where,

$$a_{\alpha}(s,x,Y_0,\phi) = \int_{0}^{s} \int_{0}^{s} \cdots \int_{0}^{s} \{ (F(t, t-s_{\alpha})L_{\phi}F(t-s_{\alpha}, t-s_{\alpha-1})L_{\phi}F(t-s_{\alpha-1}, t-s_{\alpha-2}) \}

\cdots \}

\{ (F(t-s_2, t-s_1)L_{\phi}F(t-s_1, t-s)) \}\{ \phi(x)ds_1 \cdots ds_{\alpha}$$

for $0 \leq s_{\alpha} \leq s_{\alpha-1} \leq \cdots \leq s_1 \leq 0$, and the remainder term,

$$F_{N+1}(s,x,Y_0) = \int_{0}^{s} \int_{0}^{s} \cdots \int_{0}^{s} \{ (Q(t, t-s_{N+1})L_{\phi}F(t-s_{N+1}, t-s_{N})L_{\phi}F(t-s_{N}, t-s_{N-1}) \}

\cdots \}

\{ (Q(t-s_2, t-s_1)L_{\phi}F(t-s_1, t-s)) \}\{ \phi(x)ds_1 \cdots ds_{N+1}$$

for $0 \leq s_{N+1} \leq s_{N} \leq \cdots \leq s_1 \leq 0$.

**Proof**

We integrate each side of the equation in theorem 4 above:

For $t \geq s > 0$, we have for $0 \leq s_{1} \leq s$,

$$(3.40) \quad \int_{0}^{s} \frac{\phi_1}{2} (Q(t, t-s)F(t-s_1, t-s)\phi)(x)ds_1 \int_{0}^{s} Q(t, t-s_1)L_{\phi}F(t-s_1, t-s)\phi)ds_1$$

The L.H.S. of (3.40) is given by:

$$\int_{0}^{s} \int_{0}^{s} (Q(t, t-s)F(t-s_1, t-s)\phi)(x)ds_1 \int_{0}^{s} Q(t, t-s_1)L_{\phi}F(t-s_1, t-s)\phi)ds_1$$

Since $F(t-s_1, t-s)$ and $Q(t, t)$ are identity operators, (3.41) becomes:

$$\int_{0}^{s} (Q(t, t-s)\phi(x)) = \phi(\gamma(s)) + \int_{0}^{s} (Q(t, t-s_1)L_{\phi}F(t-s_1, t-s)\phi)(x)dx_1$$

Set,

$$\phi_1 = L_{\phi}F(t-s_1, t-s)\phi$$

Then, $\phi_1$ is smooth with compact support in $M_0$. Using (3.42), we have for $0 \leq s_2 \leq s$,

$$(3.43) \quad \int_{0}^{s_2} (Q(t, t-s_1)\phi_1)(x) = \phi_1(\gamma(s_1)) + \int_{0}^{s_2} (Q(t, t-s_2)L_{\phi}F(t-s_2, t-s_1)\phi_1)(x)dx_2$$

Consequently, (3.42) becomes:

$$(3.44) \quad \int_{0}^{s} (Q(t, t-s)\phi)(x) = \phi(\gamma(s)) + \int_{0}^{s} (Q(t, t-s_1)L_{\phi}F(t-s_1, t-s)\phi)(x)dx_1$$

$$+ \int_{0}^{s} \int_{0}^{s} (Q(t, t-s_2)L_{\phi}F(t-s_2, t-s_1)L_{\phi}F(t-s_1, t-s)\phi)(x)dx_1 dx_2$$

The theorem is thus proved for $N = 1$. The rest of the proof is carried out by induction, noting that:

$$F_m(s,x,Y_0,\phi) = a_m(s,x,Y_0,\phi) + F_{m+1}(s,x,Y_0,\phi).$$
Corollary 1

\[ E(\phi(x'(s \wedge \zeta)) \exp\left\{ \int_0^s \frac{L}{V} (x'(u \wedge \zeta)) \, du \right\} ) \]

\[ = \phi(\gamma(s)) + \sum_{\alpha=1}^{N} b_{\alpha}(s, x, y_0, \phi) + F_{N+1}(s, x, y_0, \phi) \]

where \( E_\zeta \) is the mathematical expectation relative to the Wiener measure and \( \zeta \) is the first exit time of the submanifold bridge process from the tubular neighborhood \( M_0 \).

**Proof**

The proof is a direct consequence of the Theorems 1 and 5.

We now come to a key theorem of this work:

**Theorem 6 (Exact Expansion Formula)**

Let \( \gamma \) be the unique minimal geodesic from \( x \in M_0 \) to \( y_0 \) in time \( t \). Then for \( \frac{L}{V} \) and \( L_{\phi} F(1 - r_{\alpha}, 1 - r_{\alpha-1}) L_{\phi} F(1 - r_{\alpha-1}, 1 - r_{\alpha-2}) \)

\[ \ldots L_{\phi} F(1 - r_2, 1 - r_1) L_{\phi} F(1 - r_1, 1 - r_0) \]

\( \phi \) bounded, where \( L_{\phi} \phi \) is defined in (3.38), we have:

\[ k_1(x, y_0) \phi(y_0) = q_1(x, y_0) \left[ \phi(\gamma(t)) + \sum_{\alpha=1}^{N} b_{\alpha}(x, y_0, \phi) t^\alpha + R_{N+1}(t, x, y_0, \phi) t^{N+1} \right] \]

and we can therefore write:

\[ k_1(x, y_0) = q_1(x, y_0) \left[ \tau_{x,y_0} \phi(x) + \sum_{\alpha=1}^{N} b_{\alpha}(x, y_0, \phi) t^\alpha + R_{N+1}(t, x, y_0, \phi) t^{N+1} \right] \]

where \( \tau_{x,y_0} : V_x \rightarrow V_{y_0} \) is the parallel translation along the geodesic \( \gamma \) from \( x \) to \( y_0 \) in time \( t \).

If the remainder term \( R_{N+1}(t, x, y_0, \phi) \) is bounded, where for \( 1 \geq r_1 \geq r_2 \geq \ldots \geq r_N \geq r_{N+1}, \ldots (1-r_{N+1}) \geq (1-r_N) \geq (1-r_{N-1}) \geq (1-r_{N-2}) \)

\[ \ldots (1-r_2) \geq (1-r_1) \geq 0 \), we have:

\[ b_1(x, y_0, \phi) = b_1(x, y_0, \phi(y_0)) = \int_0^1 \{ (F(1,1-r_1) L_{\phi} F(1 - r_1, 0) \phi) \} (x) \, dr_1 \]

and for, \( 2 \leq \alpha \leq N \),

\[ b_{\alpha}(x, y_0, \phi) = b_{\alpha}(x, y_0, \phi(y_0)) = \int_0^1 \int_0^{r_1} \ldots \int_0^{r_{\alpha-1}} \{ (F(1,1-r_\alpha) L_{\phi} F(1 - r_\alpha, 1 - r_{\alpha-1}) L_{\phi} F(1 - r_{\alpha-1}, 1 - r_{\alpha-2}) \}

\[ \ldots L_{\phi} F(1 - r_2, 1 - r_1) L_{\phi} F(1 - r_1, 0) \phi \} (x) \, dr_1 \ldots dr_\alpha \]

\[ R_{N+1}(t, x, y_0, \phi) = \int_0^1 \int_0^{r_1} \ldots \int_0^{r_N} (Q(t, x, y_{N+1}) \{ L_{\phi} F(1 - r_N, 1 - r_{N-1}) L_{\phi} F(1 - r_{N-1}, 1 - r_{N-2}) \}

\[ \ldots L_{\phi} F(1 - r_2, 1 - r_1) L_{\phi} F(1 - r_1, 0) \phi \} (x) \, dr_1 \ldots dr_{N+1} \]

**Proofs**

(i)

First we show that:

\[ F(t, s) = F(1, \frac{s}{t}) \]

By the definition:

\[ (F(t, s) \phi)(x) = \phi(\gamma(t - s)) \]

where \( \gamma \) is the unique minimal geodesic from \( x \) to \( y_0 \) in time \( t \).

Thus, \( \gamma(s) = x + \frac{s}{t} (y_0 - x) \)

Hence,

\[ (3.45) \quad \gamma(t - s) = x + \frac{t-s}{t} (y_0 - x) = x + (1 - \frac{t}{s})(y_0 - x) \]
Now,

\[
(F(1,\frac{1}{t})\phi)(x) = \phi(\eta(1 - \frac{r}{t}))
\]

where \( \eta \) is the unique minimal geodesic from \( x \) to \( y_0 \) in time 1.

In normal coordinates,

\[
\eta(s) = x + s(y_0 - x)
\]

Hence,

\[
\eta(1 - \frac{r}{t}) = x + (1 - \frac{r}{t})(y_0 - x)
\]

Then by (3.45) and (3.46) we see that:

\[
\gamma(t-s) = \eta(1 - \frac{r}{t})
\]

Consequently,

\[
(F(t,s)\phi)(x) = \phi(\gamma(t-s)) = \phi(\eta(1 - \frac{r}{t})) = (F(1,\frac{r}{t})\phi)(x)
\]

i.e.

\[
F(t,s) = F(1,\frac{r}{t})
\]

Setting:

\[ s = tr_0, \quad s_i = tr_i, \text{ for } 1 \leq i \leq N+1, \]

we have:

\[
F(t,t-s_\alpha) = F(t,t-tr_\alpha) = F(1,\frac{t-tr_\alpha}{t}) = F(1,1-r_\alpha)
\]

and,

\[
F(t-s_\alpha,t-s_\alpha-1) = F(1,\frac{t-s_\alpha}{t-r_\alpha}) = F(1,\frac{t-r_\alpha}{t-r_\alpha}) = F(1,1-r_\alpha-1)
\]

Consequently for \( 1 \geq r_0 \geq r_2 \geq r_N \geq r_{N+1}, \) we have:

\[
\alpha_0(s, x, y_0, \phi)
= \int_0^{\tau_0} \int_0^{\tau_1} \cdots \int_0^{\tau_{N-1}} (F(1,1-r_\alpha)L_\phi F(1-r_\alpha, 1-r_\alpha-1)L_\phi F(1-r_\alpha-1, 1-r_\alpha-2) \cdots L_\phi F(1-r_2, 1-r_1)L_\phi F(1-r_1, 1-r_0)\phi)(x)dr_1 \cdots dr_\alpha
\]

The same change of variables apply to the remainder term \( R_{N+1}(t, x, y_0) \).

Consequently the "horde" of derivatives:

\[
\{F(1,1-r_\alpha)L_\phi F(1-r_\alpha, 1-r_\alpha-1)L_\phi F(1-r_\alpha-1, 1-r_\alpha-2) \cdots L_\phi F(1-r_2, 1-r_1)L_\phi F(1-r_1, 1-r_0)\phi\}(x)
\]

are continuous in \( r_0 \).

We obtain the result by taking limits as \( s \uparrow t, \) which is the same as \( r_0 \uparrow 1, \)

(given the change of variable: \( s = tr_0 \) on both sides of (3.39) in Theorem 5).

We have, as \( r_0 \uparrow 1, \)

\[
(F(1-r_1, 1-r_0)\phi)(.) = \phi(\gamma_{1-r_1}(r_0-r_1)) \rightarrow \phi(\gamma_{1-r_1}(1-r_1)) = (F(1-r_1, 0)\phi)(.)
\]

We then see from Theorem 5 that:

\[
\alpha_0(s, x, y_0, \phi) \rightarrow b_0(x, y_0, \phi)
\]

\[
= \int_0^{\tau_0} \int_0^{\tau_1} \cdots \int_0^{\tau_{N-1}} (F(1,1-r_\alpha)L_\phi F(1-r_\alpha, 1-r_\alpha-1)L_\phi F(1-r_\alpha-1, 1-r_\alpha-2) \cdots L_\phi F(1-r_2, 1-r_1)L_\phi F(1-r_1, 1-r_0)\phi)(x)dr_1 \cdots dr_\alpha
\]

The same limit procedure applies to \( F_{N+1}(t, x, y_0, \phi) \) in order to obtain \( R_{N+1}(t, x, y_0, \phi) \):

(ii) \( R_{N+1}(t, x, y_0, \phi) \)

\[
= \int_0^{\tau_0} \int_0^{\tau_1} \cdots \int_0^{\tau_N} (Q(t,tr_{N+1})L_\phi F(1-r_{N+1}, 1-r_N)L_\phi F(1-r_N, 1-r_{N-1}) \cdots L_\phi F(1-r_2, 1-r_1)L_\phi F(1-r_1, 0)\phi)(x)dr_1 \cdots dr_{N+1}
\]

where \( \zeta \) is the first exit time from the tubular neighborhood \( M_0 \) of the bridge process.

We obtain an asymptotic expansion given the boundedness assumption and this is so if the integrand above is bounded and in turn, this is so if both \( f \) and \( \Psi \) together with their derivatives, are bounded since, there is a "horde" of such derivatives involved in the remainder term.
18

\( R_{N+1}(t, x, y_0, \phi) \).

We note that since \( \phi(x) \in V_x \) and \( \phi(y_0) \in V_{y_0} \), we have:

\[ \phi(\gamma(t)) = \phi(y_0) = \tau_{xy_0} \phi(x). \]

Consequently, \( b_0(x, y_0, \phi) = \tau_{xy_0} \phi(x) \)

\[ \text{Theorem 7 (Asymptotic Expansion Formula)} \]

Under the boundedness assumption on the remainder term \( R_{N+1}(t, x, y_0, \phi) \) we have the asymptotic expansion of the heat kernel:

\[ k_t(x, y_0) = q_t(x, y_0) \left[ \tau_{xy_0} \phi(x) + \sum_{\alpha=1}^{N} b_\alpha(x, y_0) t^\alpha + o(t^N) \right] \]

and we can write,

\[ k_t(x, y_0) = q_t(x, y_0) \left[ \tau_{xy_0} + \sum_{\alpha=1}^{N} b_\alpha(x, y_0) t^\alpha + o(t^N) \right] \]

\[ \text{Proof} \]

The proof is immediate.

Since \( b_\alpha(x, y_0, \phi) \) are the expansion coefficients of \( k_t(x, y_0) \phi(y_0) \) and the heat kernel \( k_t(x, y_0) \in \text{Hom}(V_{y_0}, V_x) \), then since \( \phi(y_0) \in V_{y_0} \), the coefficients \( b_\alpha(x, y_0, \phi) \) should really be written \( b_\alpha(x, y_0) \phi(y_0) \). The coefficients \( b_\alpha(x, y_0) \) are expressible in terms of the geometric invariants of the Riemannian manifold \( M \) and the vector bundle \( V \).

4. Differential Geometric Tools

4.1. Definitions and Notation. Let \((x_1, ..., x_n)\) be the normal coordinate system based at \( y_0 \).

A normal vector field \( X \) defined by:

\[ X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}, \]

where the \( a_j, s \) are constant, is called a normal Fermi vector field.

We will denote the Riemannian (Levi-Civita) connection on \( M \) by \( \nabla \).

Let \( X_{i_1}, ..., X_{i_p} \) be normal vector fields. The \( p^{th} \) covariant derivative \( \nabla^p \) is defined inductively as follows:

\[ \nabla^p X_{i_1, ..., i_p} = \nabla_{X_{i_1}}(\nabla^{p-1}_{X_{i_2}, ..., X_{i_p}}), \]

Following standard notation, we set:

(i) \( \nabla^{p}_{i_1, ..., i_p} = \nabla^p X_{i_1, ..., i_p} \)
(ii) \( R_{\alpha \beta \gamma \lambda} = \) components of the curvature tensor of the Riemannian manifold \( M \).
(iii) \( g_{\alpha \beta} = \sum_{\gamma=1}^{n} R_{\alpha \gamma \beta \gamma} = \) components of the Ricci curvature tensor of \( M \).
(iv) \[ \tau = \sum_{\alpha, \beta = 1}^{n} R_{\alpha\beta\alpha\beta} = \sum_{\alpha=1}^{n} \theta_{\alpha\alpha} \]

denotes the scalar curvature of M.

(v)

\[ \|\theta\|^2 = \sum_{\alpha, \beta = 1}^{n} (\theta_{\alpha\beta})^2 \]

(vi) \[ \|R\|^2 = \sum_{\alpha, \beta, \gamma, \lambda = 1}^{n} (R_{\alpha\beta\gamma\lambda})^2 \]

4.2. Preliminary Geometric Expansions. In this section we give expansions in normal coordinates of the components of the metric tensor \( g_{\alpha\beta}, \alpha, \beta = 1, \ldots, n \) as well as that of the volume change factor \( \psi_{x_0} = \sqrt{\text{det}(g_{\alpha\beta})} \). These expansions will be used to compute the expansion coefficients of the heat kernel.

The needed expansions of \( g_{\alpha\beta} \) and \( \psi_{x_0} \) are abundant in the literature. See for example:

(Gray [1], Gray [2], Gray [18], Ii [19], McKean-Singer [6], Sakai [20]). We will use those of Alfred Gray in [1] and [2] because they have been carried out up to the fifth and sixth terms respectfully.

**Proposition 1** (Gray [1], Gray [2])

For \( \alpha, \beta = q + 1, \ldots, n \),

\[
g_{\alpha\beta}(x_0) = \delta_{\alpha\beta} - \frac{1}{4} \sum_{i,j=1}^{n} R_{i\alpha j\beta}(y_0)x_i x_j - \frac{1}{6} \sum_{i,j,k=1}^{n} \nabla_i R_{j\alpha k\beta}(y_0)x_i x_j x_k
\]

\[ + \frac{1}{900} \sum_{i,j,k,l=1}^{n} (-18\nabla^2_{ij} R_{k\alpha l\beta} + 16 \sum_{s=1}^{n} R_{i\alpha jk\beta}(y_0)x_i x_j x_k)
\]

\[ + \frac{1}{90} \sum_{i,j,k,h=1}^{n} \left\{ -\nabla^3_{ijk} R_{l\alpha h\beta} + 2 \sum_{s=1}^{n} \left( \nabla_i R_{jaks\beta\beta\beta} + \nabla_i R_{j\alpha k\beta\beta\beta} \right) \right\}(y_0)x_i x_j x_k x_h
\]

+ higher order terms.

Lastly we give expansion of the volume change factor:

\[ \psi_{y_0}(x) = \sqrt{\text{det}(g_{\alpha\beta}(x))} \]

The first four are given by Proposition (9.22) in [2]. The fifth and sixth terms are given on p.225 in [2].

**Proposition 2** (Gray [2])

\[ \psi_{x_0}(x) = 1 - \frac{1}{6} \sum_{i,j=1}^{n} q_{ij}(y_0)x_i x_j - \frac{1}{12} \sum_{i,j,k=1}^{n} \nabla_i q_{jk}(y_0)x_i x_j x_k
\]

\[ + \frac{1}{144} \sum_{i,j,k,l=1}^{n} \left[ \sum_{r=1}^{n} -\frac{3}{2} \nabla^2_{ij} q_{kl} + \frac{1}{2} q_{ij} q_{kl} - \frac{3}{2} R_{irjk} R_{krls}(y_0)x_i x_j x_k x_l
\]

\[ + \frac{1}{560} \sum_{i,j,k,l,h=1}^{n} \left[ -2\nabla^3_{ijk} q_{lh} + 5\nabla_i q_{jk} q_{lh} - 2 \sum_{r,s=1}^{n} \nabla_j R_{rjsx} R_{rksl}(y_0)x_i x_j x_k x_l
\]

\[ + \frac{1}{720} \sum_{i,j,k,h=1}^{n} \left\{ -\frac{5}{2} \nabla^4_{ijkh} q_{gh} + 3 \sum_{s=1}^{n} \left( \nabla^2_{ij} q_{kl} \right) q_{gh} + \frac{5}{2} (\nabla_i q_{jk})(\nabla_i q_{gh}) \right\} \]
\[ -\frac{8}{7} \sum_{a,b=1}^{n} (\nabla^2_{ij} R_{kalb}) R_{hagb} - \frac{5}{9} \theta_{ij} \theta_{kl} \theta_{hg} - \frac{16}{15} \sum_{a,b=1}^{n} (\nabla_i R_{jakb})(\nabla_i R_{hagb}) \]
\[ - \frac{16}{63} \sum_{a,b,c=1}^{n} R_{iajb} R_{kblec} R_{hagc} + \frac{2}{3} \theta_{ij} \sum_{a,b=1}^{n} R_{kalb} R_{hagb} \{ (y_0) x_i x_j x_k x_l x_g + \text{higher order terms} \}. \]

\[ \text{Proof} \]

5. Computations of the Leading Coefficients of the Expansion

Computing the coefficients at a general point \( x \in M_0 \) is intractable. We will be computing the leading coefficient at the center of normal coordinates \( x = y_0 \). We will compute \( b_\alpha(y_0, y_0) \phi(y_0) \) where \( b_\alpha(y_0, y_0) \in \text{Hom}(V_{y_0}, V_{y_0}) \) for \( \alpha = 0, 1, 2 \).

We recall that \( \Psi(x) = \Phi_{y_0}(x) \theta(x)^{-\frac{1}{2}} \) where,
\[ \Phi_{y_0}(x) = \exp \left\{ \int_0^t < b(\gamma(s)) \gamma(s) > ds \right\} \]
for a vector field \( b \) on \( M \) and \( \gamma \) a unique minimal geodesic from a generic point \( x \in M_0 \) to \( y_0 \) in time 1.

5.1. Computations of \( b_0(y_0, y_0) \phi(y_0) \) and \( b_1(y_0, y_0) \phi(y_0) \). The first two leading coefficients are very easy to compute.

**Theorem 8**

\( b_0(y_0, y_0) \) is an identity on the set of sections \( \Gamma(V) \) in the sense that:

\[ b_0(y_0, y_0) \phi(y_0) = \phi(y_0). \]

**Proof**

For \( \phi(y_0) \in V_{y_0} \)

\[ b_0(y_0, y_0) \phi(y_0) = \phi(\gamma(t)) \]
by the expansion formula. But \( \gamma(t) = y_0 \) since \( \gamma \) is the unique minimal geodesic from \( x = y_0 \) to \( y_0 \) in time t and so \( b_0(y_0, y_0) \phi(y_0) = \phi(y_0) \).

\( b_0(y_0, y_0) : V_{y_0} \rightarrow V_{y_0} \) is therefore an identity.

**Theorem 9**

\[ b_1(y_0, y_0) \phi(y_0) = \frac{1}{12} \left\{ \tau + 6W - 6 \| b \|^2 - 6 \text{div}b \right\} (y_0) \phi(y_0) + \frac{1}{2} \phi_{\dot{y}_0}(y_0) \]

**Proof**

(5.1) \( b_1(y_0, y_0) \phi(y_0) = \int_0^1 (F(1, 1-r_1)(L \dot{\gamma}(1-r_1), 0) \phi)(y_0) dr_1 \]
\[ = \int_0^1 (L \dot{\gamma}(1-r_1), 0) \phi(\gamma(r_1)) dr_1 = \int_0^1 L(\Psi F(1-r_1), 0) \phi)(y_0) dr_1 = L(\Psi \phi)(y_0) \]
\[ = \frac{1}{2} \Delta \phi(y_0) + < b(y_0), \nabla \phi(y_0) > + \frac{1}{2} \Delta \phi(y_0) + \frac{1}{2} \Delta \phi(y_0) \]
\[ + < b(y_0), \nabla \phi(y_0) > + < \nabla \phi(y_0), \nabla \phi(y_0) > \]
\[ = \frac{1}{2} \Delta \phi(y_0) + < b(y_0), \nabla \phi(y_0) > + \frac{1}{2} \Delta \phi(y_0) \phi(y_0) + \frac{1}{2} \Delta \phi(y_0) \phi(y_0) \]
\[ + < b(y_0), \nabla \phi(y_0) > + < \nabla \phi(y_0), \nabla \phi(y_0) > \]
Recall that \( \Psi = \Phi \theta^{-\frac{1}{2}} \),
where by (3.3) and (3.5),
\[ \Phi_{y_0}(x) = \exp \left\{ \int_0^1 < b(\gamma(s)) \gamma(s) > ds \right\} \]
and \( \partial_{\dot{y}_0}(x) = \sqrt{\text{det}(g_{\alpha \beta}(x))} \).
From now hence we drop the subscript \( y_0 \) from \( \Phi_{y_0} \) and \( \partial_{\dot{y}_0} \).
Recalling that $\vartheta(y_0) = 1$ and $\nabla \vartheta^{-\frac{1}{2}}(y_0) = \frac{1}{2} \frac{\partial \vartheta}{\partial r}(y_0) = 0$, we have:

$$\nabla \Psi(y_0) = \vartheta^{-\frac{1}{2}}(y_0)\nabla \Phi(y_0) + \Phi(y_0)\nabla \vartheta^{-\frac{1}{2}}(y_0) = \nabla \Phi(y_0)$$

Since,

$$\nabla \Phi(y_0) = -b(y_0)$$

by a direct computation and hence since

$$\nabla \Psi(y_0) = -b(y_0)$$

since $\vartheta(y_0) = 1$ and $\nabla \vartheta^{-\frac{1}{2}}(y_0) = 0$.

$$\Delta \vartheta^{-\frac{1}{2}}(y_0) = \frac{1}{6} \tau(y_0)$$

by direct computation.

$$\Delta \Phi(y_0) = \|b(y_0)\|^2 - div b(y_0)$$

by a direct computation.

$$\Delta \phi(y_0) = \{\phi_{i1} + W(\phi)\}(y_0)$$

by (ii) of Lemma 2 since,

$$\Lambda_i(y_0) = 0 = \Gamma_{i0j}(y_0)$$

and hence,

$$b_1(y_0,y_0)\phi(y_0)$$

(5.2) $b_1(y_0,y_0)\phi(y_0)$

$$= \frac{1}{2} \Delta \phi(y_0) + < b(y_0), \nabla \phi(y_0) > + \frac{1}{2} \Delta \Phi(y_0)\phi(y_0) + \frac{1}{2} \Delta \vartheta^{-\frac{1}{2}}(y_0) \phi(y_0)$$

$$+ < b(y_0), \nabla \Psi(y_0) > \phi(y_0) + < \nabla \Phi(y_0), \nabla \phi(y_0) >$$

$$= \frac{1}{2} \Delta \phi(y_0) + < b(y_0), \nabla \phi(y_0) > + \frac{1}{2} \left\{ \|b(y_0)\|^2 - div b(y_0) \right\} \phi(y_0) + \frac{1}{12} \tau(y_0) \phi(y_0)$$

$$+ < b(y_0), -b(y_0) > \phi(y_0) + < -b(y_0), \nabla \phi(y_0) >$$

$$= \frac{1}{2} \left\{ \phi_{i1} + W(\phi) \right\}(y_0) - \frac{1}{2} \left\{ \|b(y_0)\|^2 + div b(y_0) \right\} \phi(y_0) + \frac{1}{12} \tau(y_0) \phi(y_0)$$

The last equality is due to the fact that in the expression before the last equality,

$$< b(y_0), \nabla \phi(y_0) > + < -b(y_0), \nabla \phi(y_0) > = 0; < b(y_0), -b(y_0) > = -\|b(y_0)\|^2$$

5.2. Computation of $b_2(y_0,y_0)\phi(y_0)$. We first give some purely computational lemmas in order to compute $b_2(y_0,y_0)\phi(y_0)$

For simplicity we will take the vector field $b \equiv 0$ hence $\Psi = \vartheta^{-\frac{1}{2}}$.

Recall that we set:

$$L_{\Psi} = \frac{1}{\Psi}$$

Further we will often use the fact that $\Psi(y_0) = 1$ and $\nabla \Psi(y_0) = 0 = \nabla \vartheta^{-\frac{1}{2}}(y_0)$.

Lemma 6

$$b_2(y_0,y_0)\phi(y_0) = \int_0^1 \int_0^{r_1} J_1 dr_1 dr_2 + \int_0^1 \int_0^{r_1} J_2 dr_1 dr_2$$

where:

(i) $J_1 = L_{\Psi}(y_0)\{L\phi(y_0) + \phi(y_0)L\Psi(y_0)\}$

(ii) $J_2 = \frac{1}{2} \left\{ \frac{d^2}{dt^2} + W \right\} L_{\Psi} \phi(y_0)$

Proof

Let $\gamma_1$ be the unique minimal geodesic from $x$ to $y_0$ in time $t$.

From the expansion theorem (Theorem 6),

$$b_2(y_0,y_0)\phi(y_0) = \int_0^1 \int_0^{r_1} F(1,1-r_2) L_{\Psi} \{F(1-r_2,1-r_1)L_{\Psi} F(1-r_1,0)\}(y_0) dr_1 dr_2$$

$$= \int_0^1 \int_0^{r_1} L_{\Psi} \{F(1-r_2,1-r_1)L_{\Psi} F(1-r_1,0)\}(\gamma_1(r_2)) dr_1 dr_2$$

where $\gamma_1$ is the unique minimal geodesic from $y_0$ to $y_0$ in time $t$.

(Since $\gamma_1$ is a geodesic it must be a constant map and hence $\gamma_1(r_2) = y_0$)

$$= \int_0^1 \int_0^{r_1} L_{\Psi} \{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0) dr_1 dr_2$$

That is,

$$b_2(y_0,y_0)\phi(y_0) = \int_0^1 \int_0^{r_1} L_{\Psi} \{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0) dr_1 dr_2$$

Now, from the definition of $L_{\Psi}$,

$$L_{\Psi} \{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0) = L\{\Psi L F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0)$$

$$= L\Psi(y_0) \{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0) + L\{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0)$$

$$+ < \nabla \Psi(y_0), \nabla \{F(1-r_2,1-r_1)L_{\Psi} \phi\}(y_0) >$$
Since $\nabla \Psi(y_0) = 0 = \nabla \phi^{-\frac{1}{2}}(y_0)$, we have:

\[
\begin{align*}
(5.4) \quad & b_2(y_0, y_0) = -\int_0^1 \int_0^r \{L\Psi(y_0), F(1 - r_2, 1 - r_1)\{L\Psi \phi\} + L\{F(1 - r_2, 1 - r_1)\{L\Psi \phi\}\} \} dr_1 dr_2 \\
& \quad + L\{F(1 - r_2, 1 - r_1)\{L\Psi \phi\}\} \} dr_1 dr_2 \\
& \quad = \{L\Psi \phi\}(y_0) = \Psi^{-1}(y_0)L\{L\Psi \phi\}(y_0) = L\{L\Psi \phi\}(y_0) \\
(5.7) \quad & = L\phi(y_0) + \phi(y_0)L\Psi(y_0) + <\nabla \Psi(y_0), \nabla \phi\} \} > = L\phi(y_0) + \phi(y_0)L\Psi(y_0)
\end{align*}
\]

Consequently by (5.3) and (5.7),

\[
\begin{align*}
(5.8) \quad & J_1 = L\Psi(y_0).F(1 - r_2, 1 - r_1)\{L\Psi \phi\}(y_0) \\
& \quad = L\Psi(y_0).\{L\phi(y_0) + \phi(y_0)L\Psi(y_0)\}
\end{align*}
\]

We now compute $J_2$:

\[
\begin{align*}
(5.9) \quad & J_2 = L\{F(1 - r_2, 1 - r_1)\{L\Psi \phi\}\}(y_0)
\end{align*}
\]

Setting $L\Psi \phi = \Theta$ then,

\[
\begin{align*}
(5.10) \quad & J_2 = L\{F(1 - r_2, 1 - r_1)\Theta\}(y_0) = L\{\Theta \circ \gamma_{1-r_2}(r_1 - r_2)\}(y_0)
\end{align*}
\]

where $\gamma_{1-r_2}$ is the minimal geodesic from $x$ to $y_0$ in time $(1 - r_2)$ and so in normal coordinates: $\gamma_{1-r_2} = \frac{1-r_2}{r_2} x$.

Since, $L = \frac{1}{2} \{ (\delta^{ij}_r \nabla_{\partial_i} \nabla_{\partial_j} - \Gamma_{ik}^r \nabla_{\partial_k} + W ) \}$, we have:

\[
L\{\Theta \circ \gamma_{1-r_2}(r_1 - r_2)\}(y_0) = \frac{1}{2} \{ (\delta^{ij}_r \nabla_{\partial_i} \nabla_{\partial_j} - \Gamma_{ik}^r \nabla_{\partial_k} + W ) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0) = \frac{1}{2} \{ (\delta^{ij}_r \nabla_{\partial_i} \nabla_{\partial_j} - \Gamma_{ik}^r \nabla_{\partial_k} + W ) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0) = \delta^{ij}_r \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0)
\]

Now, by (i) of lemma 2 above,

\[
\nabla_{\partial_i} \nabla_{\partial_j} \Theta = \left\{ \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial A_i}{\partial x_j} + \Lambda_j \frac{\partial}{\partial x_j} + \Lambda_i \frac{\partial}{\partial x_j} + \Lambda_j \right\} \Theta
\]

and so,

\[
(5.11) \quad J_2 = \frac{1}{2} \{ (\delta^{ij}_r \nabla_{\partial_i} \nabla_{\partial_j} - \Gamma_{ik}^r \nabla_{\partial_k} + W ) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0) = \frac{1}{2} \{ (\delta^{ij}_r \nabla_{\partial_i} \nabla_{\partial_j} - \Gamma_{ik}^r \nabla_{\partial_k} + W ) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0)
\]

Now,

\[
\begin{align*}
\frac{\partial }{\partial x_i} \{ \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0) = \frac{\partial }{\partial x_i} F(1 - r_2, 1 - r_1)(\frac{\partial }{\partial x_i} \Theta)(x) \frac{1-r_1}{1-r_2}
\end{align*}
\]

and repeating the process:

\[
\begin{align*}
\frac{\partial^2 }{\partial x^2_i} \{ \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \} \Theta \circ \gamma_{1-r_2}(r_1 - r_2) \}(y_0) &= \frac{\partial^2 }{\partial x^2_i} \{ F(1 - r_2, 1 - r_1)(\frac{\partial }{\partial x_i} \Theta)(y_0) \frac{1-r_1}{1-r_2}
\end{align*}
\]

\[
= \{ F(1 - r_2, 1 - r_1) \frac{\partial^2 }{\partial x^2_i} \Theta)(y_0) \frac{1-r_1}{1-r_2} \} + \{ F(1 - r_2, 1 - r_1) \frac{\partial }{\partial x_i} \Theta)(y_0) \frac{1-r_1}{1-r_2} \}^2
\]

\[
= \frac{\partial^2 }{\partial x^2_i} \Theta)(y_0) \frac{1-r_1}{1-r_2} \}^2 \text{ since } \gamma_{1-r_2}(r_1 - r_2) = y_0
\]

Consequently,

\[
(5.12) \quad J_2 = \frac{1}{2} \left\{ \frac{\partial^2 }{\partial x^2_i} \Theta)(y_0) \frac{1-r_1}{1-r_2} \}^2 + 2A_i(y_0)(\frac{\partial }{\partial x_i} \Theta)(y_0) \frac{1-r_1}{1-r_2} \} \right\} + \frac{1}{2} \left\{ \frac{\partial A_i}{\partial x_i}(y_0) \Theta(y_0) + A_i^2(y_0) \Theta(y_0) + W(y_0) \Theta(y_0) \right\}
\]
\(-\frac{1}{2} \left\{ \Gamma''_{ij}(y_0) \left( \frac{\partial}{\partial x_j} \Theta(y_0) \right) + \Lambda_j(y_0) \Theta(y_0) \right\} \)

We first note that the Christoffel symbols \( \Gamma^k_{ij} \) disappear at the center of normal coordinates and so \( \Gamma''_{ij}(y_0) = 0 \).

Since we have chosen a normalized frame on \( V \), we have by Lemma (2.1.1) of [4], \( \Lambda_j(y_0) = 0 \). Further since \( \Omega_{ij} \) is skew-symmetric in \( (i,j) \), we have \( \Omega_{ii} = 0 \).

Hence by (i) of Lemma 1 above,
\[
\frac{\partial \Lambda_i}{\partial x_i}(y_0) = \frac{1}{2} \Omega_{ii}(y_0) = 0
\]
and so the above expression becomes:
\[
(5.12)(b) \quad J_2 = \frac{1}{2} \left\{ \left( \frac{\partial^2}{\partial x_j^2} \Theta(y_0) \right) \left( \frac{1}{1 - r_2^2} \right)^2 + W(y_0) \Theta(y_0) \right\}
\]
\[
= \frac{1}{2} \left\{ \left( \frac{1}{1 - r_2^2} \right)^2 \frac{\partial^2}{\partial x_j^2} + W \right\} \Theta(y_0)
\]
\[
= \frac{1}{2} \left\{ \left( \frac{1}{1 - r_2^2} \right)^2 \frac{\partial^2}{\partial x_j^2} + W \right\} L \Phi \phi(y_0)
\]

\[\blacksquare\]

**Lemma 7**

\( b_2(y_0, y_0) \phi(y_0) = I_1 + I_2 = I_1 + A + B \)

where,

(i) \( I_1 = \int_0^1 \int_0^{r_1} J_1 dr_1 dr_2 = \frac{1}{12} \Delta \Psi(y_0) \left\{ \frac{1}{2} \Delta \nabla \Psi(y_0) \phi(y_0) + \frac{1}{2} \Delta \phi(y_0) \right\} \)

(ii) \( I_2 = \int_0^1 \int_0^{r_1} J_2 dr_1 dr_2 = \frac{1}{12} W(y_0) \left\{ \left( \frac{\partial^2}{\partial x_j^2} \phi \right) + \Delta \phi \right\}(y_0) \)

where,

\[ A = \frac{1}{12} W(y_0) \left\{ \Delta \phi + \frac{1}{2} \Delta \nabla \phi \right\}(y_0) \]

\[ B = \frac{1}{24} \frac{\partial^2}{\partial x_j^2} \left( \Delta \phi \right)(y_0) + \left[ \frac{\partial^2}{\partial x_j^2} \left( \frac{1}{2} \nabla \nabla \phi \right) \right](y_0) \]

Proofs

Since,
\[
\int_0^1 \int_0^{r_1} \left( \frac{1}{1 - r_2^2} \right)^2 dr_1 dr_2 = \frac{1}{6}
\]
\[
\int_0^1 \int_0^{r_1} dr_1 dr_2 = \frac{1}{2}
\]

we have:

(i) By (i) of Lemma 6,
\[
I_1 = \int_0^1 \int_0^{r_1} J_1 dr_1 dr_2 = \frac{1}{12} L \Psi(y_0) \left\{ L \phi(y_0) + \phi(y_0) L \Psi(y_0) \right\}
\]
\[
= \frac{1}{12} \Delta \Psi(y_0) \left\{ \frac{1}{2} \Delta \nabla \Psi(y_0) \phi(y_0) + \frac{1}{2} \Delta \phi(y_0) \right\}
\]
\[
= \frac{1}{12} \Delta \Psi(y_0) \left\{ \Delta \nabla \Psi(y_0) \phi(y_0) + \Delta \phi(y_0) \right\}
\]

(ii) By (ii) of Lemma 6,
\[
I_2 = \int_0^1 \int_0^{r_1} J_2 dr_1 dr_2 = \frac{1}{2} \int_0^1 \int_0^{r_1} \left\{ \left( \frac{1}{1 - r_2^2} \right)^2 + W \right\} L \Phi \phi(y_0)
\]
\[
= \frac{1}{2} \left\{ \frac{1}{6} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} W \right\} L \Phi \phi(y_0)
\]
\[
= \frac{1}{2} \left\{ \frac{1}{6} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} W \right\} \left( \frac{1}{2} \Delta \nabla \Psi(y_0) \phi(y_0) + \Delta \phi(y_0) \right) = A + B
\]

where,
\[ A = \frac{1}{12} W(y_0) L(\Psi \phi)(y_0) \] and \[ B = \frac{1}{12} \frac{\partial^2}{\partial x_j^2} \left[ \frac{1}{2} \Delta \nabla \Psi \phi \right](y_0) \]

We now simplify \( A \) and \( B \) to get the expressions required in the lemma:

Since \( L \Psi = \frac{1}{2} \Delta \nabla \Psi, \) \( L \phi = \frac{1}{2} \Delta \phi; \) \( \Psi(y_0) = \phi(y_0) = \phi(y_0) \) and \( \nabla \phi = \phi(y_0) \)
we have:
\[
A = \frac{1}{8} W(y_0) L(\Psi \phi)(y_0) = \frac{1}{4} W(y_0) \left\{ \frac{1}{2} \Delta \phi + \frac{1}{2} (\Delta^o \vartheta^{-\frac{1}{2}}) \phi + < \nabla \Psi, \nabla \phi \right\}(y_0)
\]
\[
= \frac{1}{8} W(y_0) \left\{ \frac{1}{2} \Delta \phi + \frac{1}{2} (\Delta^o \vartheta^{-\frac{1}{2}}) \phi \right\}(y_0)
\]
(5.13) \quad = \frac{1}{8} W(y_0) \left\{ \Delta \phi + (\Delta^o \vartheta^{-\frac{1}{2}}) \phi \right\}(y_0)

We next simplify B:
\[
B = \frac{\partial^2}{\partial x^2} (\Psi^{-\frac{1}{2}} L(\Psi \phi))(y_0) = \frac{1}{12} \frac{\partial^2}{\partial x^2} \left[ (\vartheta^2 L(\vartheta^{-\frac{1}{2}}) \phi) \right](y_0) \quad = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left[ (\vartheta^2 \Delta (\vartheta^{-\frac{1}{2}}) \phi) \right](y_0)
\]
\[
= \frac{1}{23} \frac{\partial^2}{\partial x^2} \left\{ \Delta \phi + (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) + \vartheta^2 (\vartheta^2 g^{jk}) (\vartheta^2 \phi) \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]
(5.14) \quad B = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) \right\}(y_0)

\[\]
\[
+ \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\vartheta^2 g^{jk}) (\vartheta^2 \phi) \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]
\]

**Lemma 8**
\[
b_2(y_0, y_0) \phi(y_0) = I_1 + I_2 + I_3 + A + B_1 + B_2 + B_3
\]
where,
\[
I_1 = \frac{1}{4} \Delta \Psi \psi(y_0) \left\{ \frac{1}{2} \Delta \vartheta^{-\frac{1}{2}}(y_0) \phi(y_0) + \frac{1}{2} \Delta \phi \right\}(y_0)
\]
\[
A = \frac{1}{8} W(y_0) \left\{ \Delta \phi + (\Delta^o \vartheta^{-\frac{1}{2}}) \phi \right\}(y_0)
\]
and,
\[
B_1 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ g^{jk} (\nabla_{\partial_j} \nabla_{\partial_k} \phi - G^{jk}_{\partial \partial} \nabla_{\partial_j} \phi) + W(\phi) \right\}(y_0)
\]
\[
B_2 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) \right\}(y_0)
\]
\[
= \frac{1}{24} \left\{ \frac{\partial^2}{\partial x^2} (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) + (\Delta^o \vartheta^{-\frac{1}{2}}) \vartheta^2 \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]
\[
B_3 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\vartheta^2 g^{jk}) \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]

**Proofs**

B_1, B_2 and B_3 are obtained from:
\[
B = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ \Delta \phi \right\}(y_0) + \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) \right\}(y_0)
\]
\[
+ \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\vartheta^2 g^{jk}) \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]

as follows:
\[
B_1 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ \Delta \phi \right\}(y_0)
\]
\[
B_2 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\Delta^o \vartheta^{-\frac{1}{2}}) (\vartheta^2 \phi) \right\}(y_0)
\]
\[
B_3 = \frac{1}{24} \frac{\partial^2}{\partial x^2} \left\{ (\vartheta^2 g^{jk}) \left( \frac{\partial^2}{\partial x_j^2} \nabla_{\partial_k} \phi + \frac{\partial^2}{\partial x_k^2} \nabla_{\partial_j} \phi \right) \right\}(y_0)
\]

The expressions for I_1, I_2, A, B_1, B_2, and B_3 have been computed (see full version of the paper posted on my web page: http://www.facstaffwebs.umes.edu/ndnmn) in

terms of the local curvature invariants of M and V. The expansion coefficient
\[
b_2(y_0, y_0) \phi(y_0)
\]
is thus expressed in terms of the curvature invariants of the Riemannian
manifold M, the curvature invariants of the vector bundle V, and the vector bundle section
Lemma 9

We write a more simplified expression for $b_2(y_0, y_0)\phi(y_0)$:

\[ b_2(y_0, y_0)\phi(y_0) = I_1 + I_2 = I_1 + A + B = I_1 + A + B_{12} + B_{13} + B_{23} + B_{3} \]

\[ = I_1 + A + B_{12} + B_{23} + B_{3} = I_1 + A + B_{12} + B_{23} + B_{3} \]

1. \( I_1\) starts

1. \( I_1\) ends

2. \( A\) starts

2. \( A\) ends

3. \( B_{12}\) starts

3. \( B_{12}\) ends

Lemma 10

We write a more simplified expression for $b_2(y_0, y_0)\phi(y_0)$:
Heat Trace Asymptotics given, as for example in other works. Consequently in order to make comparisons with previous results on (3.3.1), we will make a final simplification.

Our expressions of the expansion coefficients of the heat kernel here are more general than usually given by other authors. Ours here clearly bring out the role of 1

\[ \sum_{j,k=1}^{n} \psi_{ij}(\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2} \Omega_{ij,\phi}) \] (y_0) \]

\[ \frac{1}{8} \sum_{j,k=1}^{n} \psi_{ij}(\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2} \Omega_{ij,\phi}) \phi(y_0) \] (12) B_3 ends

Two more terms cancel out: \[ \frac{10}{864} \tau^2 \] in B_21 and \[ \frac{1}{8} \sum_{j,k=1}^{n} \psi_{ij}(\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2} \Omega_{ij,\phi}) \] in B_22

The final expression for \( b_2(y_0, y_0) \phi(y_0) \) is thus given by:

**Theorem 10**

\[ b_2(y_0, y_0) \phi(y_0) = \frac{1}{1440} [5 \tau^2 + 2 \| R \|^2 + 12 \Delta \tau - 2 \| \theta \|^2 + 60 \tau W + 180 W^2 + 60 W_{ii} + 30 \Omega_{ij,\phi}] (y_0) \phi(y_0) + \frac{10}{864} \tau^2 \] B_21 (simplified)

Our expressions of the expansion coefficients of the heat kernel here are more general than usually given by other authors. Ours here clearly bring out the role played by the vector bundle section \( \phi \) and its derivatives. These are usually absent in other works. Consequently in order to make comparisons with previous results on Heat Trace Asymptotics given, as for example in [4] theorem (4.1.6) and [5] theorem (3.3.1), we will make a final simplification.
This is achieved by assuming that the smooth section $\phi$ of the vector bundle $V$ is constant in a small neighborhood of $y_0$ in the sense that if $\{\mu_1, \ldots, \mu_m\}$ is a local frame for $V$ at $y_0$, then in a small neighborhood of $y_0$,

$$\phi = \sum_{j=1}^m f_j \mu_j$$

where $f_1, \ldots, f_m$ are constant functions. Therefore,

$$d\phi = \sum_{j=1}^m (df_j \mu_j + f d\mu_j) = 0.$$

The last equality is due to the fact that: $df_j = 0$ since $f_j$ is a constant and $d\mu_j = 0$ by the fact that $r$ is a metric connection (see e.g. p.111 of [11]).

$$d\phi = \sum_{j=1}^m \frac{\partial \phi}{\partial x_i} dx_i = 0$$

and hence $\frac{\partial \phi}{\partial x_i} = 0$ in a small neighborhood of $y_0$, and so all partial derivatives of $\phi$ at $y_0$ are zero. We delete all derivatives of $\phi$ in Theorem 9 and Theorem 10 to have theorems 11 and 12 below:

**Theorem 11**

$$b_1(y_0, y_0) \phi(y_0) = \frac{1}{12} \tau(y_0) \phi(y_0) + \frac{1}{2} (W \phi)(y_0) - \frac{1}{2} \left\{ ||b(y_0)||^2 + divb(y_0) \right\} \phi(y_0).$$

**Remark 1**

Without the presence of the vector field terms $b_1(y_0, y_0)$ is $[a_1]$ of (34) in [3].

Without the presence of the vector field terms $b_1(y_0, y_0)$ is $e_2(x, D)$ in Theorem (4.1.6) of [4] and $a_2(F, D)$ in Theorem (3.3.1) of [5].

With the presence of the vector field terms $b_1(y_0, y_0)$ is (1) of section 4 in [6].

**Theorem 12**

$$b_2(y_0, y_0) \phi(y_0) = \frac{1}{1440} \left[ 5\tau^2 + 2 \|R\|^2 + 12\Delta \tau - 2 \|\phi\|^2 + 60\tau W + 180W^2 + 60W_{ii} + 30\Omega_{ij} \Omega_{ij} \right](y_0) \phi(y_0).$$

**Remark 2**

$b_2(y_0, y_0)$ here is $[a_2]$ of (34) in [3] (Ivan Avramidi), and is $e_4(x, D)$ in Theorem (4.1.6) of [4] (Peter Gilkey), and $a_4(F, D)$ in Theorem (3.3.1) of [5] (Peter Gilkey). Finally $b_2(y_0, y_0)$ is to be compared to (1) of section 7 of [6].

**Remark 3**

The extra factor of $\frac{1}{4}$ in $b_1(y_0, y_0) \phi(y_0)$ and $\frac{1}{4}$ in $b_2(y_0, y_0) \phi(y_0)$ here are due to the fact that our Laplace-Type operator $L = \frac{1}{2} \Delta$ has an extra factor of $\frac{1}{2}$.

6. **References**

7. N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, 2004