Sparsity Aware Nonlinear Multichannel Ultrasonic Tomographic Imaging

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Abstract—This paper presents an accelerate projected steepest descent method for nonlinear ultrasonic tomographic imaging with sparsity constraints in a multiple-input multiple-output configuration. The proposed method introduces the emerging MIMO signal processing techniques and sparsity constrained image reconstruction methods in the traditional computational imaging field, thus significantly improves the speed of image formation compared with conventional imaging method while achieving high quality images. Using numerical examples, we demonstrate the success of the algorithm.

I. INTRODUCTION

Wave propagation based tomographic imaging has wide applications in medical imaging, radar and sonar. Mathematically, the ultrasonic tomographic imaging problem is modeled by the inverse problem of a two-dimensional acoustical wave propagation equation (see e.g. [1]). In [2], we studied the tomographic imaging problem in a multiple-input multiple-output (MIMO) configuration in which multiple sources were simultaneously excited. Accordingly, the wave propagation equation is described as follows when the field of interest from the $m$-th viewing angle:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2(x)\Delta u(x,t) + \sum_{l=1}^{J_m} s(x, s_l, t)$$

where $u(x,t)$ is defined over the domain $\Omega \times [0, T]$ and the spatial variable $x = (x_1, x_2) \in \Omega$ is defined in the imaging domain. The term

$$c(x) = c_0 \sqrt{1 + f(x)}$$

is the spatial distribution of the wave speed, which depends on the acoustical potential function $f(x)$. Thus, the tomographic imaging problems amounts to find the function $f(x)$ given measured received sensor data and the governing wave equation (1). In what follows, we omit the argument $x$ for simplicity purposes. Furthermore, the wave propagation equation in (1) can be written as a nonlinear operator equation

$$\mathcal{R}_m(f) = g_m$$

where the operator

$$\mathcal{R}_m : L^2(\Omega) \rightarrow L^2(\Omega \times [0, T])$$

is a nonlinear mapping from the Hilbert spaces $L^2(\Omega)$ onto $L^2(\Omega \times [0, T])$ with respect to $f \in L^2(\Omega)$. The data $g_m$ is the measured data collected at sensors in the perimeter of the field of interest due to the source excitation from the $m$-th viewing angle. The problem of interest is to find the solution $f_*$ to (3). In general, the wave physics based tomographic imaging problem given in (1) and (3) is formulated as an inverse problem and can be solved using iterative methods. One of the commonly used iterative approaches is the adjoint method, which employs the update formula for $f$ as follows:

$$f^{k+1} = f^k + \omega \mathcal{R}_m^*(g_m - \mathcal{R}_m(f^k))$$

where $k = 0, 1, 2, \cdots$, and $\omega$ is the relaxation factor. $\mathcal{R}_m^*(\cdot)$ is the Fréchet derivative of $\mathcal{R}_m(\cdot)$ and $\mathcal{R}_m^*(\cdot)^*$ denotes the adjoint operator of $\mathcal{R}_m^*(\cdot)$. It is important to note that the iterative method (5) is often called Landweber iteration or a variation of Kaczmarz iteration [3]. By (5), our method that utilizes MIMO excitation and reception described in [2], [4] has proven to achieve much faster convergence than conventional method (see e.g. [5]) where excitation is conducted one sensor at a time. We emphasize that the MIMO method is an emerging techniques in signal processing that aims to utilize waveform diversity or spatial diversity. By applying the MIMO method in the traditional computational imaging field, we investigate problems related to how the transmission strategies affect the imaging speed and image quality.

Note that in many practical situations, a large amount of sensor data are acquired for image formation while Landweber and Kaczmarz type of iterations are rather slow, which limits its real time application. The goal of this paper is to further accelerate the iteration of the image reconstruction algorithm by using sparsity constraints with respect to the target of interests in the imaging field. We assume that the solution $f$ to the wave equation (3) has a sparse series expansion

$$f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda$$

with respect to a pre-assigned basis $\{\phi_\lambda\}$, i.e. the series expansion of $f$ has only a very small number of non-vanishing coefficients $c_\lambda$. Thus, instead of updating the function $f$...
directly as described in (5), we will update the coefficients $c_k$. One of the updating formulae (see e.g. [6], [7]) is given by

$$c_{k+1}^\lambda = \mathcal{P}_{B_d}(c_k^\lambda + \epsilon \delta c_k^\lambda)$$

(7)

where the increment term is given by

$$\delta c_k^\lambda = \langle \delta f, \phi_\lambda \rangle$$

(8)

$$\delta f^\lambda = \mathcal{R}_{m}'(\sum_\lambda c_k^\lambda \phi_\lambda)^T(g_m - \mathcal{R}_{m}(\sum_\lambda c_k^\lambda \phi_\lambda))$$

(9)

Here $\langle \cdot, \cdot \rangle$ is the inner product, $\omega$ is the relaxation term, and $\mathcal{P}_{B_d}(c)$ is the projection defined for any closed convex $\ell_1$ ball $B_d$ given by

$$B_d \triangleq \{ c \in L_2(\Omega); \| c \|_{\ell_1} \leq d \}$$

(10)

where $c = (c_1, c_2, \cdots, c_\lambda)$ and the norm

$$\ell_1 \triangleq \sum_{\lambda \in A} |c_\lambda|$$

(11)

The projection $\mathcal{P}_{B_d}(c)$ can be chosen as follows: If $\| c \|_{\ell_1} > d$, then the $\ell_2$ projection of $c$ on the $\ell_1$ ball with radius $d$ is given by

$$\mathcal{P}_{B_d}(c) = \mathcal{S}_\mu(c)$$

(12)

where $\mu$ (depending on $c$ and $d$) is chosen such that $\| \mathcal{S}_\mu(c) \|_{\ell_1} = d$. If $\| c \|_{\ell_1} \leq d$, then

$$\mathcal{P}_{B_d}(c) = \mathcal{S}_0(c) = c$$

(13)

(see also Lemma 2 in [6]), where $\mathcal{S}_\mu(c)$ is a thresholding function (see also [6])

$$\mathcal{S}_\mu(c) = \begin{cases} 
  c - \mu & \text{if } c > \mu \\
  0 & \text{if } |c| \leq \mu \\
  c + \mu & \text{if } c < \mu 
\end{cases}$$

(14)

What $\mathcal{P}_{B_d}(c)$ or $\mathcal{S}_\mu(c)$ does is to force the coefficients $c$ to be constrained within the $\ell_1$ ball of the radius $d \triangleq \| c \|_{\ell_1}$ where

$$\bar{c} = \lim_{k \to \infty} c_k$$

(15)

In other words, if the a point $c$ falls outside of the ball $B_d$, $\mathcal{P}_{B_d}(c)$ maps this $c$ into a unique point in the ball $B_d$ for which the $\ell_2$ distance to the point $c$ is minimal. The purpose of this mapping to avoid overshoot in the iteration due to the introduction of $\ell_1$ sparsity constraint to the imaging problem. According to [6], [7], the iteration (7) is considered as the solution to the approximation of the minimization problem

$$\min_{c \in B_d} \| \mathcal{R}_{m}(\sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda) - g_m \|^2$$

(16)

Hence, (16) seeks a solution that solves the nonlinear inverse problem subject to the $\ell_1$ sparsity constraint. The iteration (7) is often called projected steepest descent method [6], [7].

One should note that the assumption given in (6) requires a priori knowledge of the basis function $\phi_\lambda$. For problems of tomographic imaging of extended targets, this implies that the shape and the location of the target is known a priori, except for the acoustical potential value $f$ associated with the target. We stress that for practical problems it is still possible to obtain a rough estimate of the shape and location of the targets of interest. Under this condition and if the number of targets in the imaging field is small, our goal now is to obtain the acoustical properties (i.e., $f$ values) of the targets using the a priori knowledge of the rehspace/location of the target.

There have been many research work related to the sparsity constrained projected steepest descent method. For example, the proof for convergence of this type of projected steepest descent method for linear inverse problems is provided in [6]. Numerical verification for the projected gradient method using an explicitly formulated nonlinear operator is given in [7]. However, for the problem of wave propagation based nonlinear tomographic imaging given in (1) or (3), the nonlinear operator $\mathcal{R}_{m}$ is implicit, which means that the Fréchet derivative $\mathcal{R}_{m}'$ and the nonlinear operator $\mathcal{R}_{m}$ itself that will be needed to implement the algorithm do not have explicit expressions. As a result, it becomes challenging to develop suitable algorithm for (7) that will converge and lead to meaningful solutions. To the best of our knowledge, there is limited work for sparsity constrained nonlinear inverse problem for imaging applications. In this paper, we show how we develop the accelerated projected steepest descent method for our MIMO tomographic imaging problem to achieve improved fast image reconstruction while still obtaining high quality images. The remainder of the paper is organized as follows. The nonlinear MIMO tomographic imaging algorithm that incorporates sparsity constrains is provided in section 2. Computer simulation results are presented in section 3 for the proposed imaging algorithm.

II. THE NONLINEAR TOMOGRAPHY ALGORITHM

A. Description of the Algorithm

To develop the imaging algorithm, we will follow the projected steepest descent algorithm given in [7] to solve for an inverse problem subject to sparsity constrains where the basis function is known a-priori. The difference between the algorithm given in [7] and our MIMO imaging tomographic problem is that the nonlinear operator in [7] is explicit while in our imaging problem, the nonlinear operator $\mathcal{R}_{m}$ is implicit without having a closed expression, thus new mathematical derivation from the wave propagation perspective must be provided to develop the algorithm. This algorithm can be briefly described:

*Step 0 Initiation*: Given the operator $\mathcal{R}_{m}(c)$, its derivative $\mathcal{R}_{m}'(c)$, the measured sensor data $g_m$, the initial guess $c^0$ and the sparsity constraint $\ell_1$-ball $B_d$, the following parameter will be computed:

$$r^0 = \max \left\{ 2 \sup_{c^0 \in B_d} \| \mathcal{R}_{m}'(c^0) \|^2, 2L \sqrt{\mathcal{D}(c^0)} \right\}$$

(17)

where $L$ the Lipschitz constant. $\mathcal{D}(c^0) = \| \mathcal{R}_{m}(c^0) - g_m \|^2$. Next, the iteration for $k = 0, 1, 2, \cdots$ is performed until a pre-assigned precision/maximum number of iterations is reached.
Step 1: Calculating $\beta^k$

$$\beta^k = \max \left\{ \frac{\sup_{c^k \in B_d} \| R_m'(c^k) \|^2}{L \sqrt{D(c^k)}} \sqrt{\frac{D(c^k)}{\mathcal{D}(c^k)}} \right\}$$

(18)

Step 2: Updating $c^{k+1}$

$$c^{k+1} = \mathcal{P}_{B_d}(c^k + \frac{\beta^k}{r} R_m'(c^k)^* (g - R_m(c^k)))$$

(19)

by fixed point iteration

Step 3: Verification Check the following condition

$$\beta^k \| R_m(c^{k+1}) - R_m(c^k) \|^2 \leq \frac{r}{2} \| c^{k+1} - c^k \|^2$$

(20)

If (20) is satisfied, we increase $k$ by 1 and go to Step 1, otherwise we set $\beta^k = q \beta^k$ and go to Step 2, where the constant $q = 0.9$ is chosen here.

B. Implementation of the Algorithm

As we can see, to implement the proposed algorithm, we will need to calculate explicitly the $\| R_m(c) \|$, the Lipschitz constant $L$. Next, we compute explicitly the increment value $\delta f$ for the problem (1) where multiple sources are simultaneously excited. Let $u + \delta u$ be a solution of (1) given the acoustical potential value $f + \delta f$. Note that here we omit the $(x, t)$ for $u + \delta u$ and the superscript $(k)$ for $f + \delta f$, respectively for the purposes of simplicity. Thus we immediately obtain (see [4] for detailed derivation)

$$c_0^2 \delta f \Delta u = \frac{\partial^2}{\partial t^2} \delta u - c_0^2 (1 + f) \Delta \delta u$$

(21)

The boundary and initial conditions for (21) are as follows:

$$\frac{\partial \delta u}{\partial n} = 0, \quad \text{on} \quad \Gamma \times [0, T],$$

$$\delta u = 0, \quad \frac{\partial \delta u}{\partial t} = 0, \quad \text{on} \quad \Omega \times \{0\}$$

(22)

(23)

Employing the Taylor expansion for the non-linear operator $R_m'(f)$, we obtain

$$R_m'(f) \delta f \big|_{\Gamma \times [0, T]} = R_m(f^{(k)}) + \delta f \big|_{\Gamma \times [0, T]} - R_m(f^{(k)}) \big|_{\Gamma \times [0, T]} = \delta u \big|_{\Gamma \times [0, T]}$$

(24)

(25)

which leads to

$$\| R_m'(f) \| = \sup_{\delta f \neq 0} \frac{\| R_m'(f) \delta f \|}{\| \delta f \|} = \sup_{\delta f \neq 0} \| \delta u \|$$

(26)

Next, let $L$ be the Lipschitz constant of

$$\| R_m'(f_1) - R_m'(f_2) \| \leq L \| f_1 - f_2 \|$$

(27)

where $L$ is defined by

$$L \triangleq \sup \| R_m''(f) \|$$

(28)

For small $\delta f$, we obtain

$$\| R_m''(f) \| = \frac{\| R_m'(f + \delta f) - R_m'(f) \|}{\| \delta f \|} = \frac{\| R_m'(f + \delta f) \delta f - R_m'(f) \delta f \|}{\| \delta f \|^2}$$

(29)

$$= \frac{\| \delta u(f + h) - \delta u(f) \|}{\| \delta u \|^2} \leq \frac{2}{\| \delta u \|^2}$$

Therefore we let $L = 2 \sup_{\delta f \neq 0} \| \delta u \|$. Next, if $k \geq 1$

$$r_k = \max \left\{ \sup_{c^k \in B_R} \| R_m'(c^k) \|^2, 2L \sqrt{D(c^k)} \right\}$$

(30)

$$= \max \left\{ \sup_{\delta f \neq 0} \| \delta u \|^2, 4 \sup_{\delta f \neq 0} \| \delta u \| \sqrt{D(f^0)} \right\}$$

$$= \frac{2 \| \delta u \|^2}{\| \delta f \|^2} \sup_{\delta f \neq 0} \left\{ \| \delta u \|, 2 \sqrt{D(f^0)} \right\}$$

(31)

Similarly, we obtain

$$\beta^k = \frac{1}{2 \sqrt{D(f^k)}} \max_{\delta f \neq 0} \left\{ \| \delta u \|, 2 \sqrt{D(f^0)} \right\}$$

(32)

Therefore using (31) and (32), we obtain

$$\frac{\beta^k}{r_k} = \frac{\| \delta f \|^2}{4 \| \delta u \|^2 D(f^k)}$$

(33)

where $\delta u$ is solution of equation (21)-(23) under each iteration. Note that $\delta f$ is an arbitrary change of $f$. Next we let $\delta f$ equal to

$$\delta f^{(k)} = f^{(k)} - f^{(k-1)}$$

$$D(f^{(k)}) = g_m - R_m(f^{(k)})$$

(34)

(35)

Note that at the initial step when $k = 0$, $\delta u = 0$, so we let $\frac{\delta u}{\tau} = 1$.

III. Numerical Examples

In this section, we conduct numerical experiments to verify our results using a test phantom shown in Fig. 1 that consists of 3 extended dish-shaped targets located at $(0.04, 0.04)$ m, $(0, 0)$ m, and $(-0.04, -0.04)$ m with radius of 0.003 m, 0.005 m, and 0.004 m, respectively. The targets are sparsely imbedded in a square shaped imaging region with side length 0.16 m (i.e., $(-0.08 \text{ m}, 0.08 \text{ m})$). The transducers are placed on the four sides of the imaging region. The computational region is a square region with side length 0.2 m (i.e., $(-0.1 \text{ m}, 0.1 \text{ m})$). The acoustical potential for all the targets is chosen $f(x) = 0.06$. We call this problem as sparse imaging in that only a small portion of imaging region has potential value that reaches 0.06, for the rest of the region $f(x) = 0$. We choose a Gaussian modulated cosine function as the excitation signal

$$q(t) = e^{-t^2/(2\tau^2)} \cos(\omega_c t)$$

(36)

where $\tau = \pi/\omega_c$. The ambient speed of the sound $c_0 = 1500 m/s$. The operating frequency is at 125 kHz. The imaging
Fig. 1. (a) Ground truth of the imaging problem that consists of 3 extended targets. 640 receivers/transmitters are placed uniformly on the four sides of the square imaging region. (b) Imaging results by the accelerated MIMO using the sparsity constraints. (c) Imaging results without using sparsity constraints.

area is discretized on a $200 \times 200$ grid. A total of $160 \times 4$ transducers are uniformly placed on the four sides of the square region. Each transducer also serves as the source for excitation. Fig. 1(a) is the ground truth for the imaging problem. Fig. 1(b) depicts the image obtained using the proposed accelerated MIMO reconstruction algorithm subject to the sparsity constraints. A total of 192 iterations are performed and the reconstructed $f(x)$ reaches 0.06. Fig. 1(c) depicts the MIMO without the sparsity constraints. A total of 320 iterations are performed but the reconstructed $f(x)$ only reaches 0.04. Fig. 2 depicts the comparison of the convergence trend of the two MIMO reconstruction algorithms with and without the sparsity constraint. The convergence history is obtained by comparing the reconstructed image with the ground truth image and calculating the residual image in a mean-squared error sense. The results show that using sparsity constraint, the mean squared error residual decreases very fast, which implies that imaging algorithm using sparsity constraint converges much faster.

IV. CONCLUSIONS

We developed an accelerated projected steepest descent method for nonlinear ultrasonic tomographic imaging under a MIMO configuration using sparsity constraints. We demonstrate by numerical simulation that by adding sparsity constraints to the problem, further acceleration of the iteration for our MIMO tomographic imaging algorithm is achieved. Future research will focus on how to obtain the sparsity information during the image reconstruction process through a possible iterative learning method.

REFERENCES